Complex Valued Analytic Torsion for Flat Bundles and for Holomorphic Bundles with (1,1) Connections

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ABSTRACT
The work of Ray and Singer which introduced analytic torsion, a kind of determinant of the Laplacian operator in topological and holomorphic settings, is naturally generalized in both settings. The couplings are extended in a direct way in the topological setting to general flat bundles and in the holomorphic setting to bundles with (1,1) connections, which using the Newlander-Nirenberg Theorem are seen to be the bundles with both holomorphic and anti-holomorphic structures. The resulting natural generalizations of Laplacians are not always self-adjoint and the corresponding generalizations of analytic torsions are thus not always real-valued. The Cheeger-Muller theorem, on equivalence in a topological setting of analytic torsion to classical topological torsion, generalizes to this complex-valued torsion. On the algebraic side the methods introduced include a notion of torsion associated to a complex equipped with both boundary and coboundry maps.

1 Introduction:

In this paper the celebrated work of Ray and Singer [30, 31] on analytic torsion for Riemannian and complex Hermitian manifolds is generalized to two natural geometrical setting, one topological and the other holomorphic. In their works, coupling geometric operators, the Riemannian Laplacian and $\bar{\partial}$–Laplacian, respectively, with flat unitary bundles E yielded self–adjoint operators with real eigenvalues whose spectral properties were encoded by these two analytic torsions. These are hence real numbers, in the acyclic cases considered by Ray and Singer and are expressed as elements of real determinant line bundles.

Here the analytic torsion for the Riemannian case is extended to coupling with a general flat bundle; in this general setting the torsion has a complex

valued character. For the complex Hermitian case, the holomorphic torsion is extended to coupling with an arbitrary holomorphic bundle with compatible connection of type $(1,1)$. This includes both unitary and flat (not necessarily unitary) bundles couplngs as special cases. As will be seen, by the Newlander-Nirenberg theorem [27], and discussed in §3 below, such bundles E are characterized by being endowed with both a holomorphic structure and also an anti-holomorphic structure. However, in the present general setting the associated operators are not necessarily self–adjoint and the torsion is complex valued. The present developments involves introducing algebraically a torion invariant associated to a complex equipped with both boundary and coboundary maps.

In the flat unitary Riemannian case, Ray and Singer proved the invariance of their analytic torsion under changes of the Riemannian metric and offered ample evidence for their foundational conjecture that their analytic torsion was computable as a topological invariant, the Reidemeister-Franz torsion [30] for odd dimensional manifolds. This conjecture of Ray and Singer was proved by Cheeger and Müller [19, 25] by different methods of independent interest. In the flat unitary holomorphic case, Ray and Singer obtained results about the variation of their holomorphic torsion with respect to the change of Hermitian metric of the underlying complex manifold [31].

Here the analogues of results of Ray and Singer are proved in both our non–self adjoint settings and the extension of the Cheeger-Müller theorem is also proved.

This paper provides a general (and direct) answer to questions raised by Burghelea and Haller [14, 15] on defining torsion, algebraic and analytic, for flat bundles. They have pursued a different approach [16, 17, 18] in the topological setting, under a mild restriction on the flat bundle, following a suggestion of Müller to define an analytic torsion using a different generalization of the Laplace operator from that utilized here. Their method employs a choice of non–singular bilinear form on the flat bundle E and then an anomaly cancelation term to correct for this choice. Their conjectured relation to topological invariants was proved by Su and Zhang [35] by extending previous methods of Bismut and Zhang [8]. The seminal paper of Bismut and Zhang [8] provided the first direct proof of the Cheeger-Müller theorem and its extension to some non-unitary contexts. Underlying this approach was the remarkable deformation of the Laplacian, invented by Witten [38].

The method of proof of theorem 9.1 below generalizing the Cheeger–Müller theorem proceeds via an adaptation of these techniques used by Su and Zhang [35] in the Burghelea and Haller setting to our slightly different operator.

More closely related to the topological part of the present work, is deep work of Braverman and Kappeller [9, 10, 11, 12, 13] which also addresses a definition of torsion for non-unitary flat complex bundles over a smooth odd dimensional, oriented manifold. They achieved this, under a mild restriction on the flat bundle, using a sophisticated analysis via the odd signature operator; the general treatment in this paper goes directly via the de Rham operator. In particular, the invariant here relates to the square of the Braverman-Kappeller invariant. In recent papers [12, 13], they related the various approaches and

prove under some mild restriction a version of theorem 9.1 and so proved with this mild restriction what was a conjecture in an earlier circulated version of the present work.

As recalled in section 2, there is a quite general way of coupling any differential operator D with any flat vector bundle E, yielding a differential operator $D^\flat_E$. No metric on E is necessary for this procedure. Applied to self–adjoint operators, the resulting operators, although in general not self–adjoint, retain most of the desirable properties of that simpler case; so natural results about geometric operators should have corresponding interesting extensions to this case.

This method is applied in this paper to the Riemannian Laplacian and the result is called the flat Laplacian ${\Delta^\flat}_E$. For flat unitary bundles E , one reacquires the coupled self-adjoint operators considered by Ray and Singer [30].

As explained in section 3, in the setting of complex Hermitian manifolds, there is a further extension coupling the $\bar\partial$–Laplacian to any holomorphic bundle E endowed with a compatible type $(1,1)$ connection D, that is a connection which is compatible with the holomorphic structure and whose curvature has type $(1,1)$. Equivalently, by the Newlander Nirenberg theorem [27], the bundle E is endowed with the structure of both a holomorphic structure and also an anti-holomorphic structure. The result is called $\Delta_{E,\bar\partial}$; its definition does not use any choice of Hermitian metric on E. For E flat holomorphic it reduces to the flat coupling above. For E with connection compatible with a given Hermitian inner product on E, $\Delta_{E,\bar\partial}$ recovers the self–adjoint operators considered by Bismut, Gillet, Lebeau, Soulé [4, 5, 6, 7]. For E flat unitary, one reacquires the self-adjoint d-bar operator of Ray and Singer [31].

In section 4, the definition of the holomorphic torsion of a holomorphic bundle E with compatible type $(1,1)$ connection D is given and the local variation formula for changing Hermitian metrics is stated together with consequences parallel to the theorems of Ray and Singer for holomorphic torsion. The analogous definitions and results for the flat Riemannian case appear in section 7.

In the complex Hermitian setting of a holomorphic bundle with a compatible type $(1,1)$ connection D defined over a complex Hermitian manifold W of complex dimension n, for each p, $1 \le p \le n$, the p dimensional holomorphic torsion $\tau_{holo,p}(W,E)$ will be a non-vanishing element of a product of complex determinant line bundles:

$$\tau_{holo,p}(W,E) \ \in \{ \ det \ H^{p,*}_{\bar\partial_E}(W,E) \otimes \ [det \ H^{*;n-p}_{D''}(W,E)]^{(-1)^{n+1}} \}.$$

Note that these Dolbeault type cohomologies do not depend on a choice of Hermitian inner product g on TW. In the case that these cohomologies vanish, the acyclic case, this defines a complex number.

Quillen had introduced the determinant line bundles into the study of torsion and interpreted the Ray-Singer torsion as a metric on this line bundle [28,

29]. Here, in this general setting, we employ the complex determinant line bundles but dispense with the metric on them. The torsion is a section of the determinant line bundle, and in particular in the acyclic case is a complex number. From the present point of view, a Quillen metric arises by declaring this section to have length one.

Correspondingly, the total torsion $\tau(M,F)$ for a smooth flat bundle F over a closed smooth manifold M is in this approach a non–zero element of the determinant line bundle:

$$\tau(M,F) \in det(H^*(M,F \oplus F^*))$$

where $F^* = Hom(F,C)$ is the dual of F.

As explained in section 8, for any compact smooth, oriented manifold M, possibly with boundary, with flat complex bundle F over M, there is a combinatorial Reidemeister–Franz torsion that takes its values in precisely this determinant line for any flat bundle F. It utilizes directly a subdivision and dual subdivision of the manifold M.

A key technical difficulty of this paper is that it is **not** sufficient to just use the generalized eigenvalues of $\Delta_F^\flat$, respectively $\Delta_{E,\bar\partial}$, to define the analytic or holomorphic torsion. The zero–modes have additional structure: in each case there are two differentials, one going up and one going down, that commute with the operator $\Delta_F^\flat$, $\Delta_{E,\bar\partial}$ respectively. One must encode this information in the definition of analytic or holomorphic torsion to get control of the changes under metric variations. The added term is an algebraic torsion associated to the zero modes with these two differentials 5; it is defined and related to the standard Reidemeister–Franz torsion of a complex with bases for the cochains.

In section 5, a description of the algebraic torsion of a finite complex with two differentials, one going up and one going down, is given. It is related to the more classical approaches and expressed in terms of eigenvalues in the acyclic cases.

More explicitly, if F is a flat bundle over a smooth manifold M, then the flat extensions d_F, d_F^* of the exterior derivative d and its adjoint d^* via Riemannian metric are operators on the smooth forms with values in F, $A^*(M,F)$. These two each have square zero and commute with the flat Laplacian $\Delta_F^\flat$. Hence, the generalized zero modes of $\Delta_F^\flat$ is endowed with two differentials. To such a structure we associate an algebraic torsion and it is incorporated into the definition of the analytic torsion. Similarly for the complex holomorphic cases, we use the added information contained in the zero modes to define the holomorphic torsion.

Since signs are a key issue, to avoid a proliferation of different conventions and facilitate comparisons, we have decided to largely adopt conventions of Braverman and Kappleller [11] and Bismut and Zhang [8]; these are in agreement but differ slightly from those of Turaev [37] and Milnor [24]. For example, in accordance with Braverman and Kappeller, we will take our basic complex to be a co-chain complex with differential of degree $+1$, (C^*, d), $d : C^q \to C^{q+1}$, $d^2 =$

0. In section 4, the basic algebra concerns the situation where in addition there is a differential of degree -1, d^*, on this same C^*.

In section 6 the variation of the holomorphic torsion is computed via an analysis of the variation of the algebraic torsion associated to the generalized zero-modes. The parallel development for the flat Laplacian case appears in section 7

The combinatorial torsion for a general flat bundle F over a closed Riemannian manifold M is described in section 8.

The proof of the generalized Cheeger-Müller theorem, the equality of analytic and combinatorial torsions, is carred out in section 9. The method of proof is modeled on the work of Su and Zhang [35] with the slight modifications needed for our slightly different operator. As mentioned above, an analogue has been previously proved by Braverman and Kappeller [12, 13] in a slightly restrictive setting.

The idea of regarding the original Ray–Singer invariant, a real type invariant, [31], as giving a real norm on the determinant bundles originated in Quillen's work [28],[29] and has been extensively utilized in the seminal work of Bismut and his collaborators, [3, 4, 5, 6, 7, 8]. It is natural to conjecture analogues to the results in those papers and others in the present settings, i.e., dropping the Hermitian metrics and systematically obtaining complex valued formulae. Indeed, the naturality and good fit of the analytic details in the present use of $(1, 1)$ connections, suggests that it may provide natural settings for some extensions of notions of global analysis, symplectic geometry, and gauge theory.

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2 Coupling operators to a flat bundle:

There is a simple general principle: If D is a linear differential operator mapping smooth sections of a complex bundle E_1 to smooth sections of a complex bundle E_2 over a smooth manifold M, i.e.,

$$D: \ \Gamma(E_1) \to \Gamma(E_2), \tag{2.1 }$$

then for any complex flat bundle E there is a canonically determined associated differential operator $D^\flat_E$ from smooth sections of the tensor product $E_1 \otimes E$ to smooth sections of $E_2 \otimes E$

$$D^\flat_E : \Gamma(E_1 \otimes E) \to \Gamma(E_2 \otimes E) \tag{2.2 }$$

This is specified as follows:

Since a differential operator is defined locally, we need but specify $D^\flat_E$ over an open set, say $U \subset M$ over which the flat bundle E is trivial. Let the restriction

$E|U$ have a basis of flat smooth sections $s_1, s_2, \cdots, s_k$, with k the dimension of the bundle E. Then define

$$(D^\flat_E|U): \ \Gamma(E_1 \otimes E|U) \to \Gamma(E_2 \otimes E|U) \tag{2.3 }$$

via $\Sigma_{i=1}^k \ f_i \otimes s_i \mapsto \Sigma_{i=1}^k \ D(f_i) \otimes s_i$ for any smooth sections $f_i, i = 1, \cdots, k$ of $E_1|U$. Since the transition mappings comparing flat sections are constant and D is linear, these local definitions are compatible and define the desired operator $D^\flat_E$.

Proceeding thusly for any elliptic operator, say D, the resultant operator has symbol $\sigma(D^\flat_E)$ equal to the symbol of D tensor the identity on E, so the operator $D^\flat_E$ is elliptic. The primary difference is that for general flat E, one is led from self-adjoint elliptic operators to non-self adjoint ones.

Applied to the Laplace operator, Δ, of a Riemannian manifold M, this construction produces the flat Laplacian, $\Delta^\flat_E : A^*(M, E) \to A^*(M, E)$, acting on smooth forms with values in the flat bundle E. In the case that E is unitary, this flat Laplacian will be the standard self-adjoint extension of the Laplacian Δ where the unitary structure is used to define the inner-product. However, without any choice of Hermitian metric, using just the flat structure on E, the flat Laplacian $\Delta^\flat_E$ is well-defined.

3 Holomorphic bundles with type $(1,1)$ connections and a $\bar\partial$–Laplacian

There is a natural parallel construction for the $\bar\partial$–Laplacian. It is described in this section.

Let W be a complex manifold with Hermitian inner product, say $g =< .. >$ and $E \to W$ be a complex holomorphic, bundle over W endowed with a linear connection D.

Let $\bar\partial : A^{p,q}(W, C) \to A^{p,q+1}(W, C), \partial : A^{p,q}(W, C) \to A^{p+1,q}(W, C)$ be the standard operators obtained by decomposing by type the exterior derivative

$$d = \bar\partial + \partial$$

acting on complex valued smooth forms of type (p, q). By $d^2 = 0$, one has $(\bar\partial)^2 = 0, \ (\partial)^2 = 0$. Let $< ., . >$ be the induced Hermitian metric on these (p, q) forms and $\bar\partial^*$ denote the adjoint of $\bar\partial$ under this Hermitian metric on TW

$$< \bar\partial s_1, \ s_2 >=< s_1, \ \bar\partial^* s_2 > \ \ for \ s_1 \in A^{p,q-1}(W, C) \ and \ s_2 \in A^{p,q}(W, C)$$

here $\bar\partial^* : A^{p,q}(W, C) \to A^{p,q-1}(W, C)$.

By E holomorphic, the $\bar\partial$ operator on forms on W with complex values has a unique natural extension to the smooth forms with values in E,

$$\bar\partial_E : A^{p,q}(W, E) \to A^{p,q+1}(W, E).$$

This first order differential operator is uniquely characterized by:
1) $\bar{\partial}_E(f\ s) = (\bar{\partial}_E f) \wedge\ s + f(\bar{\partial}_E\ s)$ for any smooth function f; and,
2) over any open set U for which t is holomorphic section of E and any smooth (p, q) form a with complex values $\bar{\partial}_E(a \otimes t)|_U = (\bar{\partial} a) \otimes\ s|_U$.

A linear connection D on E is by definition a linear mapping of smooth sections:

$$D : \Gamma(E) \to \Gamma(T^*W \otimes_R E) = \Gamma((T^*W \otimes_R C) \otimes_C E)$$

satisfying the Leibnitz formula $D(f\ s) = df \otimes s + f\ D(s)$.

The complex structure on TW provides the natural splitting

$$T^*W \otimes C \cong T^{*\prime}W \oplus T^{*\prime\prime}W$$

with $T^{*\prime}W$, respectively $T^{*\prime\prime}W$, the $\mp\ i$ eigenspaces of the operator $(J \otimes Id)$ where $J : TW \to TW,\ J^2 = -Id$ specifies the complex structure. Hence, the smooth one forms with values in E, $\Gamma((T^*W \otimes_R C) \otimes_C E)$ splits as a direct sum:

$$\begin{aligned}&\Gamma((T^*W \otimes_R C) \otimes_C E)\\ &\cong A^{0,1}(W, E) \oplus A^{1,0}(W, E) \cong \Gamma(T^{*\prime}W \otimes_C E) \oplus \Gamma(T^{*\prime\prime}W \otimes_C E)\end{aligned}$$

Under this decomposition, the connection D decomposes as a sum: $D = D' \oplus D''$ with $D' : A^{0,0}(W, E) = \Gamma(E) \to A^{0,1}(W, E)$ and $D'' : A^{0,0}(W, E) = \Gamma(E) \to A^{1,0}(W, E)$.

The connection D is said to be **compatible** with the holomorphic structure on E if $D' = \bar{\partial}_E$. In particular, in this case any holomorphic section, t of E over an open set U of W, satisfies $D'(t) = \bar{\partial}_E t = 0$. The natural extension of D'' to all (p, q) forms is then $D'' := \bar{\partial}_E : A^{p,q}(W, E) \to A^{p,q+1}(W, E)$ in this case.

Now as above, the operator D'' has a unique extension:

$$D'' : A^{p,q}(W, E) \to A^{p+1,q}(W, E)$$

D'' is uniquely characterized by:
1) $D''(f\ s) = (\partial f) \wedge\ s + f(D''\ s)$ for any smooth function f; and,
2) for any smooth section t of E and any smooth (p, q) form a with complex values $D''(a \otimes t)|_U = (\partial \otimes a)\ t + (-1)^{p+q}\ a\ \wedge D''(t)$.

In the standard manner one defines the curvature of the linear connection D via

$$curvature = D^2 = (D')^2 + (D'D'' + D''D') + (D'')^2$$

This curvature is a two form with values in $End(E)$, the self maps of E, and its type decomposition $A^{0,2}(W, End(E)) \oplus A^{1,1}(W, End(E)) \oplus A^{2,0}(W, End(E))$ is $(D')^2, (D'D'' + D''D'), (D'')^2$. We say the connection is of **type** $(1, 1)$ if its curvature is of type $(1, 1)$, that is,

$$(D')^2 = 0,\ and\ (D'')^2 = 0$$

The first condition follows from $D' = \bar{\partial}_E$ when D is compatible with the holomorphic structure on E. By the Newlander–Nirenberg theorem [27, 22] the

condition $(D'')^2 = 0$ is equivalent to having a holomorphic structure on the conjugate $\bar{E}$ of E, or equivalently an anti–holomorphic structure on E.

Conversely, if E is endowed with a holomorphic structure, there is a unique linear operator $D' : \Gamma(E) \to \Gamma(T^{0,1}W \otimes E)$ with $D'(f\ s) = \overline{\partial} f \otimes\ s + f(D'\ s)$ for any smooth function f and $D'(s) = 0$ for any holomorphic section s. The condition that any point has an open neighborhood O for which E has a basis, say $\{s_i\}$ of holomorphic sections, implies that $(D')^2 = 0$, since for any smooth section s, one has $s|0 = \Sigma_i\ h_i\ s_i$ and consequently, $(D')^2(s)|0 = D'(\overline{\partial} h_i \otimes s_i) = \overline{\partial}\ \overline{\partial} h_i \otimes s_i = 0$. If in addition, E is endowed with an anti-holomorphic structure, then there is a unique linear operator $D'' : \Gamma(E) \to \Gamma(T^{1,0}W \otimes E)$ with $D''(f\ s) = (\partial f) \otimes\ s + f(D''\ s)$ for any smooth function f and as before $(D'')^2 = 0$. Together the sum $D := D' + D''$ enjoys the property $D(fs) = ((\overline{\partial} f) \otimes\ s + f(D'\ s)) + ((\partial f) \otimes\ s + f(D''\ s)) = df \otimes s + f(D(s))$. That is, D is a standard linear connection. It is of type $(1,1)$ since $(D'')^2 = (D')^2 = 0$.

These observations prove the proposition:

Proposition 3.1 *A complex bundle E endowed with a holomorphic and anti-holomorphic structure defines a unique connection D of type $(1,1)$ compatible with these two structures. Here compatible means that the type decomposition $D = D' + D''$ yields operators with $D's = 0$ for holomorphic sections and $D''s = 0$ for anti-holomorphic sections.*

Conversely, a complex bundle E endowed with a type $(1,1)$ connection has a uniquely determined holomorphic and anti-holomorphic structure.

The last statement uses the Newlander–Nirenberg theorem [27, 22].

Let D be assumed to be a linear connection which is compatible with the given holomorphic structure and having curvature of type $(1,1)$.

To specify the desired generalization of the standard d-bar Laplacian on smooth forms

$$\Delta_{\bar{\partial}} = \bar{\partial}\partial + \partial\bar{\partial} : A^{p,q}(W,C) \to A^{p,q}(W,C)$$

to the smooth (p,q) forms with values in E, $A^{p,q}(W,E)$, it is helpful give more explicit details.

Somewhat confusingly, the star operator $\star$ acting on forms is a complex conjugate linear mapping

$$\star : A^{p,q}(W,C) \to A^{n-p,n-q}(W,C) \qquad (3.4\ )$$

induced by a conjugate linear bundle isomorphism. It is characterized by the property that for $a, b \in A^{p,q}(W,C)$,

$$< a, b >= \int_W \ a \wedge \star b \qquad (3.5\ )$$

where $< a,b >= \int_W < a(x), b(x) > \ dvol_W$ with $< a(x), b(x) >$ the pairing over $x \in W$ induced by the Hermitian inner product on T_xW.

Now $\star$ composed with the natural conjugation mapping

$$conj : A^{p,q}(W,C) \to A^{q,p}(W,C) \tag{3.6 }$$

is a complex linear mapping, induced by a bundle isomorphism. Here $conj$ is induced by the bundle automorphism $T^*W \otimes_R C \to T^*W \otimes_R C,\ v \otimes \lambda \mapsto v \otimes \bar{\lambda}$ of the complexified cotangent bundle. Denote this induced, composite, complex linear mapping by $\hat{\star}$. That is,

$$\hat{\star} = conj\ \star : A^{p,q}(W,C) \to A^{n-q,n-p}(W,C) \tag{3.7 }$$

This complex linear isomorphism is induced by a complex linear mapping of bundles, so may be coupled with any bundle E via $\hat{\star} \otimes Id_E$

$$\hat{\star} \otimes Id_E : A^{p,q}(W,E) \to A^{n-q,n-p}(W,E) \tag{3.8 }$$

It should be noted that $\hat{\star}$ being complex linear may be coupled to a complex linear bundle mapping, such as the identity mapping. The bundle mapping $\star$ can be coupled only to a conjugate complex linear mapping.

Recall again that the exterior derivative d has the type decomposition $d = \bar{\partial} + \partial$ decomposition, where $\bar{\partial} : A^{p,q}(W,C) \to A^{p,q+1}(W,C)$ and $\partial : A^{p,q}(W,C) \to A^{p+1,q}(W,C)$.These correspond under conjugation

$$\partial = conj\ \bar{\partial}\ conj : A^{p,q}(W,C) \to A^{p+1,q}(W,C).$$

Recall also that the adjoint of $\bar{\partial}$ under the chosen Hermitian inner product on TW has the well known explicit description

$$\bar{\partial}^* = -\ \star \bar{\partial} \star .$$

In particular,

$$\bar{\partial}^* = -\hat{\star}\ \ conj\ \bar{\partial}\ conj\ \hat{\star} = -\hat{\star}\ \partial\ \hat{\star}.$$

Motivated by this formula, since D'' extends ∂ and $\hat{\star} \otimes Id_E$ extends $\hat{\star}$ to the sections of E, one defines

$$\bar{\partial}^*_{E,D''} = -(\hat{\star} \otimes Id_E) \cdot D'' \cdot (\hat{\star} \otimes Id_E)$$

and introduces the desired generalization of the d-bar Laplacian for E a holomorphic bundle with compatible type $(1,1)$ connection by:

$$\Delta_{E,\bar{\partial}} = \bar{\partial}_E \bar{\partial}^*_{E,D''} + \bar{\partial}^*_{E,D''} \bar{\partial}_E$$

Note that $(\bar{\partial}^*_{E,D''})^2 = 0$ since $(D'')^2 = 0$ for the given type $(1,1)$ connection D and $(\bar{\partial}_E)^2 = 0$. Also, both the differentials $\bar{\partial}^*_{E,D''}$ and $\bar{\partial}_E$ commute with $\Delta_{E,\bar{\partial}}$.

In the case that E has a flat connection, called D, utilizing the method of section 2, $\bar{\partial}^*_{E,D''}$ above is just the coupled version of $\bar{\partial}^* = -\hat{\star}\ \partial\ \hat{\star}$ and $\Delta_{E,\bar{\partial}}$ is the coupled version of the d-bar Laplacian $\Delta_{\bar{\partial}}$.

It is important to note that if a smooth Hermitian inner product is chosen on E, say $< .,. >_E$, then the adjoint of $\bar{\partial}_E$ is **not** in general the above mapping. For example, its definition does not utilize any such choice. Rather, as specified by Müller [26], the adjoint is defined in terms of the induced conjugate linear bundle isomorphism $\mu : E \to E^*$, of E and its dual E^* as:

$$adjoint(\bar{\partial}_E) = -(\star^{-1} \otimes \mu)^{-1} \; \bar{\partial}_{E^*} \quad (\star \otimes \mu) \tag{3.9 }$$

With this definition $< \bar{\partial}_E s, t >_E = < s, adjoint(\bar{\partial}_E) \; t >_E$ for the induced inner product on forms with values in E. In most of the literature, this adjoint is (rather sloppily) written as $- \star \bar{\partial} \star$. Note that the conjugate complex linear operators, $\star$ and μ appear here, not the complex linear one $\hat{\star}$. [Since both $\star$ and μ are both conjugate linear bundle isomorphisms, this makes sense.]

In the case that the holomorphic bundle E is endowed with a Hermitian inner product, say $< .,. >_E$, which is compatible with the connection D, in the sense that

$$d \; < s_1, s_2 >_E = < D \; s_1, s_2 >_E + < s_1, D \; s_2 >_E,$$

that is, the connection D is unitary, then the conjugate linear mapping μ identifies the dual bundle E^* equipped with its dual connection D^* with E equipped with its connection D. Consequently in this unitary situation, the adjoint $adjoint(\bar{\partial}_E)$ equals precisely $\bar{\partial}^*_{E,D''}$ and so the standard self-adjoint d-bar Laplacian equals the above one,

$$\Delta_{E,\mu} = (\; \bar{\partial}_E + adjoint(\bar{\partial}_E))^2 \; . = \Delta_{E,\bar{\partial}}$$

Hence, the above operator restricts to the standard self–adjoint one considered by Ray and Singer [31], Bismut, Gillet, Lebeau, Soulé in [3, 4, 5, 6, 7] for E flat unitary.

Nonetheless, in all type $(1,1)$ compatible holomorphic cases, it is apparent from these formulas that the first order differential operators $\bar{\partial}^*_{E,D''}$ and the adjoint $adjoint(\bar{\partial}_E)$ have identical symbols and thus differ by a smooth bundle automorphism, say α,

$$\bar{\partial}^*_{E,D''} = adjoint(\bar{\partial}_E) + \alpha \; .$$

Thus in all cases,

$$\Delta_{E,\bar{\partial}} = (\; \bar{\partial}_E + adjoint(\bar{\partial}_E))^2 + \alpha\bar{\partial}_E + \bar{\partial}_E\alpha \tag{3.10 }$$

is an elliptic second order partial differential equation with scalar symbol.

Since D is a type $(1,1)$ compatible connection, it follows that

$$\bar{\partial}_E : A^{p,*}(W,E) \to A^{p,*}(W,E) \; has \; (\bar{\partial}_E)^2 = 0 \tag{3.11 }$$

$$D'' : A^{*,n-p}(W,E) \to A^{*,n-p}(W,E) \; has \; (D'')^2 = 0, \tag{3.12 }$$

so two Dolbeault type homologies $H^{p,*}_{\bar{\partial}_E}(W,E)$ and $H^{*,n-p}_{D''}(W,E)$ are well defined. These cohomologies appear in the main theorem for the holomorphic torsions. The proposed invariant is to lie in the product of determinants.

$$\tau_{holo,p}(W,E) \ \in \{ \ det \ H^{p,*}_{\bar{\partial}_E}(W,E) \otimes \ [det \ H^{*,n-p}_{D''}(W,E)]^{(-1)^{n+1}} \}. \tag{3.13 }$$

It will extend the Ray-Singer holomorphic torsion [31] from their case of E unitarily flat, and so holomorphic, to this more general and flexible setting of coupling with any holomorphic bundle with compatible type $(1,1)$ connection. [For E flat holomorphic, one can replace D'' by $\partial^\flat_E$, the flat extension of ∂.]

Its variation with the choice of Hermitian metric g on TW is specified in theorem 4.4 . The cohomologies are independent of that choice. This theorem is a generalization of the Ray-Singer theorem of [31].

Foundational Example:

An important universal example of a holomorphic $(1,1)$ connection is described as follows: Let V be a complex vector space of dimension n, and $Gr_k(V)$ denote the Grassmannian of k-dimensional subspaces of V. That is, $Gr_k(V) = \{E_1 \subset V \mid dim_C \ E_1 = k\}$. Similarly, let $Gr_{n-k}(V) = \{E_2 \subset V \mid dim_C \ E_2 = (n-k)\}$, the $n-k$ dimensional subspaces. Let

$$\mathcal{O} = \{(E_1,E_2) \mid (E_1,E_2) \in \ Gr_k(V) \times Gr_{n-k}(V) \ with \ E_1 \cap E_2 = \{0\}\}$$

That is, $\mathcal{O}$ is the open subset of those pairs that are transverse, or equivalently $E_1 + E_2 \cong V$.

Over $\mathcal{O}$ one has the natural subbundle

$$\mathcal{S}_1 = \{p \in E_1 \mid (E_1,E_2) \in U\}$$

and a natural inclusion of $\mathcal{S}_1 \subset V \times Gr_k(V)$ into the trivial bundle and a natural projection defined by $V \mapsto V/E_2 \cong E_1$ of this trivial bundle $V \times Gr_k(V)$ back onto $\mathcal{S}_1$. Let these bundle mappings be denoted by

$$i : \mathcal{S}_1 \subset V \times Gr_k(V) \ and \ p : V \times Gr_k(V) \to \mathcal{S}_1$$

Hence, one may define a canonical connection D on smooth sections of $\mathcal{S}_1$ by the following procedure. For f a smooth section of $\mathcal{S}_1$, that is $f \in \Gamma(\mathcal{S}_1)$ and X a vector field on $\mathcal{O}$, use the standard derivative ∇_X on the product bundle $V \times Gr_k(V)$ applied to $i(f)$ and then project via p to $\mathcal{S}_1$ again. That is, set

$$D_X(f) = p \ \nabla_X(i(f))$$

By construction for g a smooth function on U, $D_X(g \ f) = p \ \nabla_X(i(g \ f)) = p \ \nabla_X(g \ i(f)) = p \ (g \ \nabla_X(i(f)) + (X \ g) \ (i(f))) = g \ p \ \nabla_X(i(f)) + (X \ g) \ f = g \ D_X(f) + (X \ g) \ f$, so D is a connection on the bundle $\mathcal{S}_1$ as desired.

It is claimed that the connection D is naturally a holomorphic connection of type $(1,1)$ for a suitable complex structure on $\mathcal{O}$. This is the complex structure

on $\mathcal{O}$ induced from that on the product of Grassmanians $G_k(V) \times Gr_{n-k}(V)$ where the complex structure on the first factor $Gr_k(V)$ comes from the standard complex structure $J = i$ on V and the complex structure on the second factor $Gr_{n-k}(V)$ comes from the conjugate complex structure $-J$ acting on V.

To see this concretely fix a pair (E_1, E_2) in U and take linearly independent vectors $e_1, e_2, \cdots, e_n$ where E_1 is the span of $\{e_1, \cdots, e_k\}$ and E_2 is the span of $\{e_{k+1}, \cdots, e_n\}$. Now for complex coordinates $z_{p,q},\ 1 \le p \le k, 1 \le q \le (n-k)\}$ specify the k vectors

$$s_p = e_p \ + \ \Sigma_{q=1}^{n-k} \ z_{p,q} \ e_{k+q}$$

and let $E(\{z_{p,q}\})$ be the span of these k vectors. This provides an open set, say O_1, in $Gr_k(V)$ and a complex coordinate system for the complex structure induced from J acting on V. Here E_1 is the k-plane with vanishing coordinates. Let Z denote the complex $k \times (n-k)$ matrix with entries $z_{p,q}$. Let $\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix}$ the column vector with entries the s_p's.

Similarly, for complex coordinates $u_{r,s},\ 1 \le r \le (n-k), 1 \le s \le k\}$ specify the $n-k$ vectors

$$t_r = \Sigma_{s=1}^{k} \ \overline{u_{r,s}} \ e_s \ + \ e_{k+r}$$

and let $E(\{\overline{u_{r,s}}\})$ be the span of these $n-k$ vectors. Note the conjugates in the definition of t_r. This provides an open set, say O_2, in $Gr_{n-k}(V)$ and a complex coordinate system for the complex structure induced from $-J$. Thus E_2 is the $n-k$-plane with vanishing coordinates. Let $\overline{U}$ denote the complex $(n-k) \times k$ matrix with entries $\overline{u_{r,s}}$. Let $\vec{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_{n-k} \end{pmatrix}$ the column vector with entries the t_r's.

Together the complex coordinates $z_{p,q}, u_{r,s}$ specify the desired complex structure on the open set $O_1 \times O_2$ in the product $G_k(V) \times Gr_{n-k}(V)$. The s_p's provide a basis of sections of the subbundle $\mathcal{S}_1$ while the t_r's provide as basis of sections of the subbundle $\mathcal{S}_2$.

The relation between the vectors s_p, t_r and the standard vectors basis vectors $\{e_1, \cdots, e_n\}$ is

$$\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = \begin{pmatrix} I_k & Z \\ \overline{U} & I_{n-k} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

where I_l is the identity matrix with l rows and columns. Let W denote the above $n \times n$ matrix. The condition that the subspaces $E(\{z_{p,q}\}), E(\{\overline{u_{r,s}}\})$ be transverse is precisely that $det\ W \neq 0$. That is, on $\mathcal{O} \cap (O_1 \times O_2)$ these provide coordinates. Let the inverse of the matrix W, which is defined on $\mathcal{O} \cap (O_1 \times O_2)$, be written also in block form:

$$W = \begin{pmatrix} I_k & Z \\ \overline{U} & I_{n-k} \end{pmatrix} \ and \ W^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

In these terms one computes,

$$D(\vec{s}) = p\ (\nabla\ \ i\,(\vec{s})) = p\ ((\ 0, \quad dZ\ ) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}) =$$
$$p\ ((\ 0, \quad dZ\ ) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix})$$
$$= p\ (dZ \cdot \gamma \cdot \vec{s} + dZ \cdot \delta \cdot \vec{t}) = dZ \cdot \gamma \cdot \vec{s}$$

That is, the connection form for this basis $\vec{s}$ is $\Theta = dZ \ \cdot\ \gamma$, a 1-form of type $(1,0)$.

Correspondingly, the curvature K is $K = d\Theta - \Theta \wedge \Theta$ via $D^2(\vec{s}) = D(\Theta \otimes \vec{s}) = d\ \Theta \otimes \vec{s} - \Theta \wedge\ D(\vec{s}) = (d\Theta - \Theta \wedge \Theta) \otimes \vec{s}$.

In this instance $K = d(dZ \cdot \gamma) - dZ \cdot \gamma \wedge dZ \cdot \gamma = -dZ\ \wedge d\gamma - dZ \cdot \gamma \wedge dZ \cdot \gamma = -dZ \wedge (d\gamma + \gamma \cdot dZ \cdot \gamma)$.

However, in view of the identity

$$d(W^{-1}) = \begin{pmatrix} d\alpha & d\beta \\ d\gamma & d\delta \end{pmatrix} = -W^{-1} \cdot dW \cdot W^{-1}$$
$$= -\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} 0 & d\ Z \\ \overline{d\ U} & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
$$= -\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} dZ \cdot \gamma & dZ \cdot \delta \\ \overline{d\ U}\ \cdot \alpha & \overline{d\ U} \cdot \beta \end{pmatrix}$$
$$= -\begin{pmatrix} \star & \star \\ \gamma \cdot dZ \cdot \gamma + \delta\ \overline{d\ U} \cdot \alpha & \star \end{pmatrix},$$

one obtains the equality $d\gamma = -\gamma \cdot dZ \cdot \gamma - \delta \cdot \overline{d\ U} \cdot \alpha$, so the curvature K is

$$K = +\ dZ \wedge (\delta\ \overline{d\ U} \cdot \alpha)$$

a curvature of type $(1,1)$, as expected.

Note that the sections $\vec{s}$ provide a holomorphic basis of sections since $D(\vec{s}) = dZ \cdot \gamma \cdot \vec{s}$ is of type $(1,0)$. This defines the required holomorphic structure.

Similarly, $(I - Z \cdot \gamma)\vec{s}$ provides a basis of anti-holomorphic sections, since $D((I - Z \cdot \gamma)\vec{s})$ is of type $(0,1)$ as is seen from the computation:

$$D((I - Z \cdot \gamma)\vec{s}) = -(dZ \cdot \gamma + Z \cdot d\gamma) \cdot \vec{s} + (I - Z \cdot \gamma) \cdot D(\vec{s})$$
$$= -(dZ \cdot \gamma + Z \cdot d\gamma) \cdot \vec{s} + (I - Z \cdot \gamma) \cdot (dZ \cdot \gamma \cdot \vec{s}) = -Z \cdot d\gamma \cdot \vec{s} - Z \cdot \gamma \cdot (dZ \cdot \gamma \cdot \vec{s})$$
$$= -Z \cdot (d\gamma + \gamma \cdot (dZ \cdot \gamma) \cdot \vec{s} = +\ Z \cdot (\delta \cdot \overline{d\ U} \cdot \alpha) \cdot \vec{s}$$

This defines the corresponding anti-holomorphic structure.

More conventionally, one chooses a Hermitian metric, say $< .,. >$ on V, then the subbundle, say $\mathcal{E}_1$, over the Grassmanian $Gr_k(V)$ acquires an induced Hermitian metric, so there is a unique connection D which is compatible with the metric $< .,. >$, in that

$$d < s_1, s_2 >=< Ds_1, s_2 > + < s_1, Ds_2 >$$

for any smooth sections s_1, s_2 of this subbundle. This compatible connection then has curvature of type $(1,1)$.

Using the Hermitian metric, for each k-dimensional subspace E_1 let $(E_1)^\perp = \{w \mid \ < v, w >= 0\}$ be the orthogonal complement, a $(n-k)$ dimensional subspace. Hence, the orthogonal projection of V onto E_1 is precisely the projection of V via the mapping p, $V \to V/(E_1)^\perp \cong E_1$. It is evident that for sections s_1, s_2 of $\mathcal{E}_1$ regarded as sections of the trivial bundle, i.e., as $i(s_1), i(s_2)$ for the inclusion $\mathcal{E}_1 \subset V \times Gr_k(V)$, one has the formula:

$$\begin{aligned}&X < i(s_1), i(s_2) >=< \nabla_X\ i(s_1), i(s_2) > + < i(s_1), \nabla_X\ i(s_2) > \\ &=< p\ \nabla_X\ i(s_1), s_2 > + < s_1, p\ \nabla_X\ i(s_2) >\end{aligned}$$

That is, the connection given by $f \mapsto p\ \nabla\ i(f)$ satisfies the defining property above. Consequently, one has

$$D(f) = p\ \nabla\ i(f).$$

That is, the standard compatible connection associated to the Hermitian metric $< .,. >$ is obtained in this natural fashion.

An alternative method to see this is to directly compare the respective connection 1-forms and see their equality. Given a Hermitian metric, one can define a mapping:

$$\begin{aligned}&F : Gr_k(V) \subset Gr_k(V) \times Gr_{n-k}(V) \\ &via \\ &E_1 \mapsto (E_1, (E_1)^\perp).\end{aligned}$$

Note that the image of F lies in $\mathcal{O}$. By the above, the induced connection recovers precisely the compatible connection D on $\mathcal{E}$.

4 Holomorphic torsion and Hermitian metric variation:

For E a holomorphic bundle over the complex manifold W with Hermitian metric $g =< .,. >$, let D be a linear connection on E which is compatible with the holomorphic structure and is of type $(1,1)$. Let W have complex dimension n, so real dimension $2n$.

Define the operator $\Delta_{E,\bar\partial}$ by

$$\Delta_{E,\bar\partial} = (\bar\partial_E + \bar\partial^*_{E,D''})^2 = \bar\partial_E \bar\partial^*_{E,D''} + \bar\partial^*_{E,D''} \bar\partial_E \tag{4.14 }$$

yielding a generally non-self adjoint operator, which commutes with both of $\bar\partial_E, \bar\partial^*_{E,D''}$ which have squares equal to zero.

Choose a Hermitian metric, say $< .,. >_E$ on the complex bundle E. As noted in the last section, the adjoint of ∂_E differs by a bundle mapping, say α, from $\bar\partial^*_{E,D''}$,

$$\bar\partial^*_{E,D''} = adjoint(\bar\partial_E) + \alpha \tag{4.15 }$$

and so in addition

$$\Delta_{E,\bar{\partial}} = (\ \bar{\partial}_E + adjoint(\bar{\partial}_E))^2 + \alpha\bar{\partial}_E + \bar{\partial}_E\alpha \qquad (4.16\ )$$

is an elliptic second order partial differential equation with scalar symbol.

Consequently, standard methods of elliptic theory apply, e.g., the methods of Atiyah, Patodi, Singer and Seeley, [2, 33, 34] to prove the following basic results summarized here, see section 10. In modified form they appear in the papers of Ray and Singer [30, 31].

Firstly, the spectrum of $\Delta_{E,\bar{\partial}}$ is discrete and has generalized eigenspaces of finite multiplicity.

As a second step towards understanding the spectral properties of these operators, recall an observation from Atiyah-Potodi-Singer's part III [2]. There they also study the eta invariant of the operator B_{ev} coupled to a flat non-unitary bundle. The following is just a more explicit version.

Lemma 4.1 *Fix a Hermitian inner product* $< .,. >$ *and use this to define an inner product on forms with values in* E *via* $< f, g >= \int_M \ < f(x) \wedge \star g(x) > \ dvol_x$*. Then in terms of the smooth bundle mapping* α *with* $\bar{\partial}^*_{E,D''} = adjoint(\bar{\partial}_E) + \alpha$*, the spectrum of* $(\bar{\partial}_E + \bar{\partial}^*_{E,D''})$ *lies in the strip of* $\{\lambda\}$ *satisfying*

$$-||\alpha|| \leq Im\ \lambda\ \leq +||\alpha||.$$

If $\alpha \neq 0$*, the spectrum of* $\Delta_{E,\bar{\partial}}$ *lies on or inside the parabola enclosing the positive x-axis:* $\{\lambda\}$ *with*

$$Re\ \lambda \geq \frac{(Im\ \lambda)^2}{4||\alpha||^2} - ||\alpha||^2$$

Of course, if $||\alpha|| = 0$*, then the spectrum of the operator* $\Delta_{E,\bar{\partial}} = (\ \bar{\partial}_E + adjoint(\bar{\partial}_E))^2$ *is real non-negative.*

Let $K > 0$ be a real number which is not the real part of any eigenvalue. Let $S(p, q, K)$ be a complete enumeration of all the generalized eigenvalues counted with multiplicities with real part greater than K of $\Delta_{E,\bar{\partial}}$ acting on $A^{p,q}(W, E)$. That is, $\Re(\lambda_j) > K$.

Then the zeta function

$$\zeta_{p,q,E,K,g}(s) = \Sigma_{\lambda_j \in S(p,q,K)}\ \frac{1}{\lambda_j^s}\ , \qquad (4.17\ )$$

defined using the principle values for the complex powers, converges for $Re(s) > n/2$ with $n = dim_R\ W = 2n$. If $\Pi_{E,K,g}$ denotes the spectral projection on the span of the generalized eigenvectors with generalized eigenvalues with real part less than K and $Q_{E,K,g} = (1 - \Pi_{E,K,g})$, then the elliptic operator $Q_{E,K,g}\ \Delta_{E,\bar{\partial}}$ fits the setting of Seeley [33, 34]. In particular, its complex powers $[Q_{E,K,g}\Delta_{E,\bar{\partial}}]^{-s}$ are well defined, as in that paper. Also as in [34], the "heat

kernel" $e^{-t\ Q_{E,K,g}\ \Delta_{E,\bar\partial}} = Q_{E,K,g} e^{-t\ \Delta_{E,\bar\partial}}$ is well defined for t real positive, is of trace class, and for $Re(s) > N$ the formula

$$[Q_{E,K,g}\ \Delta_{E,\bar\partial}]^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}\ e^{-t\ Q_{E,K,g}\ \Delta_{E,\bar\partial}}\ dt \tag{4.18}$$

holds and its trace gives the zeta function $\zeta_{p,q,E,K,g}(s)$ for the bundle E.

Moreover, the methods of Seeley imply that $\zeta_{p,q,E,K,g}(s)$ has a meromorphic extension to the whole complex plane and is analytic at $s = 0$. Consequently, the derivative at $s = 0$, $\zeta'_{p,q,E,K,g}(0)$ is meaningful.

Following Ray and Singer [31], one sets

$$\begin{aligned} &Ray - Singer - Term(p, E, K, g) \\ &\qquad = exp((1/2)\ \Sigma_{q=0}^{n}\ (-1)^q\ q\ \zeta'(p, q, E, K, g)(0)) \end{aligned}$$

The methods of Ray and Singer again apply to prove the following lemma about the Ray–Singer terms for E, for details see section 10. Here the metric dependence is explicitly recorded in the operators.

Lemma 4.2 *Fix $K > 0$. Let $g(u)$, $a \le u \le b$, be a smooth family of Hermitian metrics on TW such that the operators $\Delta_{E,\bar\partial,g(u)}$ have no generalized eigenvalues with real parts equal to K.*

Let the star operator $\star_{u,p,q}$ be the bundle isomorphism associated to the Hermitian inner product $g(u)$,

$$\star_{u,p,q} : \Lambda^p T^{1,0} W \otimes \Lambda^q T^{0,1} W \xrightarrow{\star_u} \Lambda^{n-p} T^{1,0} W \otimes \Lambda^{n-q} T^{0,1} W\ , \tag{4.19}$$

and similarly for $\hat\star_u$. and set $\alpha_{u,p,q} = (\star_{u,p,q})^{-1}\ d/du\ (\star_{u,p,q}) =$ $(\hat\star_{u,p,q})^{-1}\ d/du\ (\hat\star_{u,p,q})$, a the self mapping of the bundle $\Lambda^p T^{1,0} W \otimes \Lambda^q T^{0,1} W$ depending on u.

As is the case, by the results of Seeley, let the trace $Tr(\alpha_{u,p,q} e^{-t\Delta_{E,\bar\partial}})$ have an asymptotic expansion for small t given by

$$Tr(\alpha_{u,p,q} e^{-t\Delta_{E,\bar\partial}}) = \Sigma_{j=0}^{M} a_{j,u,p,q}\ t^{-N/2+j} + 0(t^{-N/2+M+1})$$

with $N = dim\ W = 2n$ and $M > N/2 + 1$. Here $a_{u,p,q}$ is given by explicit local formulas $a_{u,p,q} = \int_W\ b_{u,p,q}$ with the forms $b_{u,p,q}$ expressed directly, locally, in terms of the Hermitian metric $g(u)$, the connection form of D, and their covariant derivatives. In particular, the coefficient of t^0 is $a_{N/2,u,p,q}$.

Let $\Pi_{E,K,p,q,u}$ be the spectral projection onto the span of the generalized eigenvectors of $\Delta_{E,\bar\partial,g(u)}$ acting on (p, q) forms with generalized eigenvalues of real part less than K.

Then

$$\begin{aligned} &Ray - Singer - Term(p, E, K, g(u)) \\ &= exp((1/2)\ \Sigma_{q=0}^{n}\ (-1)^q\ q\ \zeta'(p, q, E, K, g(u))(0)) \end{aligned}$$

varies smoothly with u, for $a \le u \le b$

Moreover,

$$\begin{aligned} &d/du[log\ (\ Ray - Singer - Term(p, E, K, g(u))\ ] \\ &= (1/2)\Sigma_{q=0}^{n}\ (-1)^q\ [Tr(\Pi_{E,K,p,q,g(u)}\ (\alpha_{u,p,q} \otimes Id_E)) + \textstyle\int_W\ b_{n,u,p,q}] \end{aligned}$$

Similarly, the methods of Ray and Singer straight-forwardly give the expected dependence of the $Ray-Singer-Term(p,E,K,g)$ on changing $K>0$. It is:

Lemma 4.3 *Fix $L>K>0$. Let g be a Hermitian metric on TW such that the operator $\Delta_{E,\bar\partial,g}$ acting on $A^{p,*}(W,E)$ has no eigenvalues with real part equal to K or L. Let $\{\lambda_{p,q,j} | j=1\cdots n_q\}$ be a complete enumeration counting with multiplicities of the generalized eigenvalues of $\Delta_{E,g}$ acting on $A^{p,q}(W,E)$ which have real part in the range K to L. Then the formula below holds:*

$$\left[\frac{Ray-Singer-Term(p,E,K,g)}{Ray-Singer-Term(p,E,L,g)}\right]^2 = \left[\prod_{q=0}^{n}\left(\prod_{j=1}^{n_q}\ \lambda_{p,q,j}\right)^{(-1)^q\, q}\right]^{-1}$$

For details see section 10.

Let $C^{p,q}(E,K,g)$ denote the span in $A^{p,q}(W,E)$ of the generalized eigensolutions of $\Delta_{E,\bar\partial,g}$ with generalized eigenvalues with real part less than K. By elliptic theory, $C^{p,q}(E,K,g)$ is a finite dimensional subspace of smooth sections.

Since $\Delta_{E,\bar\partial}$ commutes with $\bar\partial_E, \bar\partial^*_{E,D''}$ which each have square zero, by D a type $(1,1)$ connection, the graded complex

$$C^{p,\star}(E,K,g) := \bigoplus_{q=0}^{n}\ C^{p,q}(E,K,g)$$

has two differentials, d, d^*, $d=\bar\partial_E$ and $d^*=\bar\partial^*_{E,D''} = -(\hat\star\otimes Id_E)D''(\hat\star\otimes Id_E)$ with $d: C^{p,q}(E,K,g) \overset{\bar\partial_E}{\to} C^{p,q+1}(E,K,g)$ increasing the grading by one and $d^*: C^{p,q}(E,K,g)\to C^{p,q-1}(E,K,g)$ decreasing degree by one. That is, one has the same complex equipped with two differentials: $C^{p,\star}(E,K,g), d, d^*$ of degrees $+1,-1$, respectively.

In the next section, §5, it will be proved that in the general algebraic setting of a graded complex of length n, C^* equipped with two differentials, d of degree $+1$ and d^* of degree -1, i.e, (C^*,d,d^*), there is a natural non-vanishing algebraic torsion invariant

$$torsion(C^*,d,d^*) \in (det\ H^*(C^*,d)) \otimes (det\ H_*(C^*,d^*))^{(-1)}\ .$$

Here $H^q(C^*,d) = (ker\ d: C^q\to C^{q+1})/(im\ d: C^{q-1}\to C^q)$, the cohomology, and $H_q(C^*,d^*) = (ker\ d^*: C^q\to C^{q-1})/(im\ d^*: C^{q+1}\to C^q)$, the homology.

Hence, one gets a non-vanishing torsion invariant

$$\begin{aligned} &torsion((C^{p,*}(E,K,g),\bar\partial,\bar\partial^*_{E,D''})) \\ &\qquad \in (det\ H^*(C^{p,*}(E,K,g),\bar\partial_E) \otimes (det\ H_*(C^{p,*}(E,K,g),\bar\partial^*_{E,D''})^{(-1)} \end{aligned}$$

These cohomologies and homologies can be identified as follows:

Let M be the multiplicity of the generalized eigenvalue 0 for $\Delta_{E,\bar\partial}$ and $\mathcal{H}$ denote the span of these generalized eigenvectors with generalized eigenvalue 0. By spectral theory $\mathcal{H}$ is finite dimensional and consists of smooth sections. Then, by adapting standard Hodge–theoretic methods, one has the direct sum decompositions:

$$\begin{aligned} A^{*,*}(W,E) = \mathcal{H} \oplus \bar\partial_E[(\Delta_{E,\bar\partial_E})^M\ A^{*,*}(W,E)] \\ \oplus((\hat\star \otimes Id_E)\partial_E(\hat\star \otimes Id_E))[(\Delta_{E,\bar\partial})^M\ A^{*,*}(W,E)] \end{aligned}$$

and $\bar\partial_E$ maps the second summand isomorphically to the first. By this means one can easily show that the inclusion of co-chain complexes

$$(C^{p,*}(E,K,g),\bar\partial_E) \subset (A^{p,*}(W,E),\bar\partial_E)$$

induces an isomorphism on cohomology. That is, $H^q(C^{p,*}(E,K,g),\bar\partial_E) \cong H^{p,q}_{\bar\partial}(W,E)$. In particular, $det\ H^*(C^{p,*}(E,K,g),\bar\partial_E) \cong det\ H^{p,*}_{\bar\partial_E}(W,E)$.

Similarly, by $(\hat\star \otimes Id_E)^2|A^{p,q}(W,E) = (-1)^{(p+q)(2n-(p+q))}$, it follows that $(\hat\star \otimes Id_E)(-(\hat\star \otimes Id_E)D''(\hat\star \otimes Id_E))(\hat\star \otimes Id_E)^{-1} = \pm\ D''$; so $(\hat\star \otimes Id_E)$ induces a complex linear isomorphism of the graded complex $(C^{n-*,n-p}(E,K,g),\pm D'')$ to the complex $(C^{p,*}(E,K,g),d^*)$ sending $C^{n-q,n-p}(W,E) \to C^{p,q}(W,E)$ This identifies $H_q(C^{p,*}(E,K,g),d^*)) = H^{n-q,n-p}_{D''}(W,E)$ and thus the determinants $det\ H_q(C^{p,*}(E,K,g),\bar\partial^*_{E,D''})) = det\ H^{n-q,n-p}_{D''}(W,E)$.

Hence, in this special situation the algebraic invariants may be regarded as an element of the same complex line,

$$\begin{aligned} torsion((C^{p,*}(E,K,g),d,d^*), \\ \in\ det\ H^{p,*}_{\bar\partial}(W,E)) \otimes [det\ H^{n-*,N-p}_{D''}(W,E)]^{-1} \\ \cong [det\ H^{p,*}_{\bar\partial}(W,E))] \otimes [det\ H^{*,n-p}_{D''}(W,E)]^{(-1)^{n+1}} \end{aligned} \tag{4.20 }$$

for any real number $K > 0$ and any choice of Hermitian metric g on TW. This last line is independent of the choice of Hermitian metric g.

Now let $L > K > 0$ be real and positive. It will follow from the algebraic lemmas B,C of §5, that under the assumptions of lemma 4.3, the algebraic torsion of $(C^{p,*}(E,K,g),d,d^*)$ and that for L with $K > L > 0$ are precisely related by the eigenvalues $\lambda_{q,j}$ of lemma 4.3 as follows:

$$\begin{aligned} torsion(C^{p,*}(E,L,g),d,d^*)) \\ = torsion(C^{p,*}(E,K,g),d,d^*)) \\ \bullet \left[\prod_{q=0}^{n} (\prod_{j=1}^{n_q}\ \lambda_{q,j})^{(-1)^q\, q}\right]^{-1} \end{aligned} \tag{4.21 }$$

as elements of $[det\ H^{p,*}_{\bar\partial_E}(W,E)] \otimes [det\ H^{*,n-p}_{D''}(W,E)]^{(-1)^{N+1}}$.

Similarly, it will follow from lemma 6.1 of §10 for a smooth family of Hermitian metrics $g(u)$ satisfying the assumptions of lemma 4.3 that one gets the equality

$$d/du \left[\; log \; torsion(C^*_p(E,K,g(u)),d,d^*)\right] \\ = - \; Tr(\Pi_{E,K,g(u)} \; (\alpha_{g(u)} \otimes Id_E)) \qquad . \tag{4.22 }$$

As an immediate consequence of the above two equations , the main theorem below follows:

Theorem 4.4 (Variation theorem of holomorphic torsion) *For any choice of Hermitian inner product g on TW, pick a real number $K > 0$ for which the operators $\Delta_{E,\bar\partial,g}$ acting on $A^{*,*}(W,E)$ have no eigenvalue with real part equal to K. Define the graded complex $C^{p,*}(E,K,g) = \bigoplus_{q=0}^N \; C^{p,q}(E,K,g)$ with its two differentials, d, d^* as above. Then the combination*

$$\tau_{holo,p}(W,E) \; := \quad [\; torsion((C^{p,*}(E,K,g),d,d^*)] \\ \bullet [Ray - Singer - Term(p,E,K,g)]^2$$

is independent of the choice of $K > 0$.

By definition $\tau_{holo,p}(W,E)$ is a non-vanishing element of the determinant line bundle

$$\tau_{holo,p}(W,E) \; \in [\; det \; H^{p,*}_{\bar\partial}(W,E)] \otimes \; [det \; H^{*,N-p}_{D''}(W,E)]^{(-1)^{N+1}}. \tag{4.23 }$$

For a smooth variation of Hermitian metrics $g(u), a \le u \le b$, $\tau_{holo,p}(W,E,g(u))$ varies smoothly. Here the explicit dependence on the metric is recorded. Also with the notation of lemma 4.2, $d/du \; \tau_{holo,p}(W,E,g(u))$ is given by the local formula:

$$d/du \; \tau_{holo,p}(W,E,g(u)) = \Sigma_{q=0}^N \; (-1)^q \; \int_W \; b_{n/2,u,p,q}$$

with $n = dim_R \; W = 2N$.

In the acyclic cases it is a complex number. In the case that E is acyclic and flat unitary, this torsion is a real number. Since the operator $\Delta_{E,\bar\partial}$ is self-adjoint in this case, one may take $K > 0$ less than the smallest non-vanishing real eigenvalue; for this choice of $K > 0$, the algebraic correction term is just $+1$ and the above reduces to the formula implicit in Ray and Singer [31].

This theorem has a corollary which generalizes the explicit theorem of Ray and Singer on holomorphic torsion [31].

Corollary 4.5 *If E is a holomorphic bundle with connection D which is compatible and type $(1,1)$ over a Hermitian complex manifold W and F_1, F_2 are two flat complex bundles over E of the same dimension, then the "quotient" of torsions*

$$\tau_{holo,p}(W,E \otimes F_1, g(u)) \otimes [\tau_{holo,p}(W,E \otimes F_2, g(u))]^{-1}$$

in the tensor product of determinant line bundles

$$\begin{aligned}&([\ det\ H^{p,*}_{\bar\partial}(W, E\otimes F_1)]\otimes\ [det\ H^{*,N-p}_{D''}(W, E\otimes F_1)]^{-1})\\ &\otimes([\ det\ H^{p,*}_{\bar\partial}(W, E\otimes F_2)]\otimes\ [det\ H^{*,N-p}_{D''}(W, E\otimes F_2)]^{-1})^{-1}\end{aligned}$$

is independent of the Hermitian metric $g(u)$ chosen.

This corollary follows since the two bundles $E\otimes F_1, E\otimes F_2$ are locally identical as bundles with connections, so the local corrections $b_{n/2,u,p,q}$ are the same and cancel in computing the variation of the holomorphic torsions.

5 An algebraic torsion for complexes with boundary and coboundary:

In this section the torsion of a chain complex C^* over the complex numbers equipped with two differentials, d going up with $d^2 = 0$, and d^* going down with $(d^*)^2 = 0$, is defined. On the one hand (C^*, d) is a cochain complex,$d : C^q \to C^{q+1}, d^2 = 0$, and also (C^*, d^*) is a chain complex, $d^* : C^q \to C^{q-1}, (d^*)^2 = 0$, so the cohomologies of d, $H^q(C^*, d)$, and the homologies of d^*, $H_q(C^*, d^*)$, can be formed. The torsion of (C^*, d, d^*) is to be an element of the product of determinant lines

$$torsion(C^*, d, d^*) \in det(H^*(C^*, d)) \otimes det(H_*(C^*, d^*))^{-1}$$

formed from the cohomology and homology of C^*.

Here a complex one dimensional vector space L is called a complex line. The dual of a complex line L is denoted by L^{-1}. This torsion will be shown to have good algebraic properties and the relation to other standard algebraic definitions is explained. It is crucial that certain sign difficulties are dealt with carefully, so the definition involves a suitable choice of signs.

For E a k dimensional vector space over the complex numbers, C, set

$$det(E) = \Lambda^k(E)$$

, the complex line given by the highest exterior product, $k = dim\ E$. By definition, if $\mathbf{v} = \{v_1, \cdots, v_k\}$ is an ordered basis of E, then the ordered wedge product $\Lambda\mathbf{v} := v_1 \wedge v_2 \wedge \cdots v_k$ is a basis for $det(E)$ and if $\mathbf{w} = \{w_1, \cdots, w_k\}$ is another ordered basis, then $\Lambda\mathbf{w} = [\mathbf{w}/\mathbf{v}] \cdot \boldsymbol{\Lambda}\mathbf{v}$ where the complex number $[\mathbf{w}/\mathbf{v}]$ is the determinant of the k by k matrix expressing the ordered basis $\mathbf{w}$ in terms of the ordered basis $\mathbf{v}$.

By convention set $det(E) = C$ for the 0-dimensional vector space $E = \{0\}$ and take its standard generator to be $1 \in C$.

Similarly, if (C^*, d) is a finite dimensional cochain complex of length N

$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \to \cdots \xrightarrow{d} C^N \to 0 \qquad (5.24\ )$$

so $d^2 = 0$, set

$$det(C^*) = \bigotimes_{q=0}^{q=N} (\; det \; C^q)^{(-1)^q} \; .$$

In this context, one defines the q-coboundries, $B^q = dC^{q-1}$, the q-cocyles, $Z^q = ker \; d : C^q \to C^{q+1}$, and the q-cohomologies $H^q(C^*, d) = Z^q/B^q$ and the also associated determinant line

$$det(H^*(C^*, d)) = \bigotimes_{q=0}^{q=N} (\; det \; H^q(C^*, d))^{(-1)^q} \; .$$

In this terminology, following the conventions of Bismut and Zhang [8], the torsion isomorphism of the graded cochain complex (C^*, d) is a standard isomorphism of determinant lines:

$$\tau : det(C^*) \stackrel{\cong}{\to} \; det(H^*(C^*, d)) \tag{5.25 }$$

recalled below.

Now if in addition, C^* has another differential $d^* : C^q \to C^{q-1}$ decreasing dimension and $(d^*)^2 = 0$,

$$0 \leftarrow C^0 \stackrel{d^*}{\leftarrow} C^1 \stackrel{d^*}{\leftarrow} \cdots \leftarrow \cdots \stackrel{d^*}{\leftarrow} C^N \leftarrow 0, \tag{5.26 }$$

then correspondingly one has a standard isomorphism of determinant lines:

$$\tau' : det(C^*) \stackrel{\cong}{\to} \; det(H_*(C^*, d^*)) \tag{5.27 }$$

defined also below. Here $H^q(C^*, d^*) = Z_q(C^*)/B_q(C^*)$ is the q-homology groups of d^*. By definition the q-cycles are $Z_q(C^*) = ker \; d^* : C^q \to C^{q-1}$ and the q-boundaries are $B_q(C^*) = d^* C^{q+1}$.

For the chain complex with these two differentials,(C^*, d, d^*), there is then defined the induced isomorphism of complex lines:

$$det(H_*(C^*, d)) \stackrel{(\tau')^{-1}}{\to} \; det(C^*) \stackrel{\tau}{\to} det(H^*(C^*, d))$$

which then defines a element of the product of determinant lines:

$$\tau(C^*, d, d^*) \in det(H^*(C^*, d)) \otimes (det(H_*(C^*, d^*)))^{-1}$$

Definition of the algebraic torsion of the complex with two differentials: (C^*, d, d^*)

One sets

$$\begin{aligned}&torsion(C^*, d, d^*)\\&= (-1)^{S(C)}\tau(C^*, d, d^*) \in det(H^*(C^*, d)) \otimes (det(H_*(C^*, d^*)))^{-1}\end{aligned}$$

Here the sign $(-1)^{S(C^*)}$ is defined by setting

$$\begin{aligned}S(C^*) = \quad &\Sigma_q\ [dim_C\ B_{q-1}(C^*) \cdot dim_C\ B^{q+1}(C^*)\\&+dim_C\ B^{q+1}(C^*) \cdot dim_C H_q(C^*, \partial)\\&+dim_C\ B_{q-1}(C^*) \cdot dim_C H^q(C^*, \partial^*)].\end{aligned} \tag{5.28 }$$

The mysterious sign is chosen so that this algebraic invariant has the desirable properties A,B,C below.

To specify the isomorphism τ for the cochain complex C^*, d, fix a direct sum decompositions of vector spaces

$$C^q = B^q \oplus H^q \oplus A^q \tag{5.29 }$$

where B^q are the coboundaries, $B^q \oplus H^q$ are the cocyles, and A^q fills out the rest of C^q. Note $A^N = \{0\}$. Then by choice H^q is naturally isomorphic to the cohomology, $H^q \cong H^q(C^*, d)$ under the projection of Z^q onto $H^q(C^*, d)$.

Taking the wedge product in this order establishes a natural isomorphism

$$\Psi : (det\ B^q) \otimes (det\ H^q) \otimes (det\ A^q) \cong det\ C^q \tag{5.30 }$$

Fix generators $c(q) \in det\ C^q, a(q) \in det\ A^q$ for each $q = 0, \cdots, N$. Use the convention that for the 0-dimensional subspace, $\{0\}$, $det\ \{0\} = C$ by definition and the chosen generator is $1 \in C$.

Let $d(a(q))$ denote the generator of $det\ B^{q+1}$ determined by the isomorphism $d : A^q \to B^{q+1}$. Then for each $q = 0, \cdots, N$ there is a unique element $h(q) \in det\ H^q$ such that

$$c(q) = \Psi(\ d(a(q-1))\ \otimes\ h(q)\ \otimes\ a(q)\ ) \tag{5.31 }$$

For $x \in L$ a non-vanishing element of the one dimensional vector space L, let x^{-1} be the unique element of $L^{-1} = L^*$ which maps x to $+1$.

With these choices, the algebraic torsion mapping

$$\tau : det(C^*) \stackrel{\cong}{\to}\ det H^*(C^*, d) \tag{5.32 }$$

is defined as the isomorphism which maps

$$c(0) \otimes c(1)^{-1} \otimes \cdots \otimes c(N)^{(-1)^N} \mapsto h(0) \otimes h(1)^{-1} \otimes \cdots \otimes h(N)^{(-1)^N}$$

It is a foundational theorem that the resultant mapping is independent of all choices [24, 8].

[Note that Braverman and Kappeller [10] introduce an additional sign at this point in their definition of a related isomorphism. For simplicity this is not done here. The cost of this simplicity is that one must introduce the total sign correction $(-1)^{S(C)}$ in the above definition of $torsion(C^*, d, d^*)$ to reestablish desirable functional properties.]

More concretely and explicitly, let $\{h^{q,k}|k = 1, \cdots, v_q\}$, $v_q = dim_C\ H^q$ be a chosen ordered basis for $H^q(C^*, d)$ for each $q = 0, \cdots, N$ and $\mathbf{c}^q = \{c^{q,1}, c^{q,2}, \cdots, c^{q,n_q}\}, n_q = dim_C\ C^q$, be a chosen ordered basis for C^q in each degree, $q = 0, \cdots, N$. Let $c(q)$ be the ordered wedge product, $\wedge_j\ c^{q,j} \in det\ C^q$, and $h(q)$ be the ordered wedge product $\wedge_k\ h^{q,k} \in det\ H^q$.

Following the notation of Ray and Singer [30], chose an ordered basis, say $\mathbf{b}^q = \{b^{q,1}, b^{q,2}, \cdots, b^{q,s_q}\}$, for the coboundaries $B^q = Im\ d : C^{q-1} \to C^q$ in C^q. Now chose $\tilde{\mathbf{b}}^q = \{\tilde{b}^{q,1}, \tilde{b}^{q,2}, \cdots, \tilde{b}^{q,s_q}\}$ in C^{q-1} so that $d\tilde{b}^{q,i} = b^{q,i}$ in B^q for $i = 1, \cdots, s_q$. Thus taking the ordered elements $\tilde{\mathbf{b}}^q$ as a basis for A^{q-1} and $\mathbf{b}^q$ as a ordered basis for B^q, one has $d(\wedge_j \tilde{b}^{q,j}) = (\wedge_j b^{q,j}) \in det\ B^q$.

Now $Z^q/B^q \cong H^q$, so pick elements $lh^{q,k} \in Z^q \subset C^q$ projecting to the basis elements $h^{q,k}$ of H^q. Taking the ordered set $\mathbf{lk}^q := \{lh^{q,k}|k = 1, \cdots, v_k\}$ as a basis for $H^q \subset C^q$ gives the above direct sum decomposition of C^q, since with these choices $\{\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}\}$ is an ordered basis for C^q. This may be compared to the chosen basis $\mathbf{c}^q$.

In this notation one has the translation of invariants:

$$\begin{aligned} &\tau(c(0) \otimes c(1)^{-1} \otimes \cdots \otimes c(N)^{(-1)^N}) \\ &= \left[\prod_{q=0}^{N} [\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{c}^q]^{(-1)^q} \right]^{-1} \\ &\times (h(0) \otimes h(1)^{-1} \otimes \cdots \otimes h(N)^{(-1)^N}) \end{aligned} \qquad (5.33\ )$$

In the below this concrete point of view is often taken. For a choice of bases for C^* and for H^* one forms the above complex number $\prod_{q=0}^{N} [\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{c}^q]^{(-1)^q}$. For the above choices one defines

$$\tau(C^*, d, \{\mathbf{h}_q\}, \{\mathbf{c}_q\}) := \left[\prod_{q=0}^{N} [\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{c}^q]^{(-1)^q} \right]^{-1} \qquad (5.34\ )$$

regarding it as specifying the isomorphism $\tau : det(C^*) \to det(H^*(C^*, d))$ above. A simple check shows $\tau(C^*, d, \{\mathbf{h}_q\}, \{\mathbf{c}_q\})$ is independent of all choices but those of the ordered bases $\{\mathbf{c}^q\}, \{\mathbf{h}^q\}$.

In analogy to the isomorphism 5.32 there is a natural isomorphism of determinant lines defined by the chain complex (C^*, d^*):

$$\tau' : det\ C^* \to det\ H_*(C^*, d^*)\ . \qquad (5.35\ )$$

It is defined as follows:

To specify the isomorphism τ' for the cochain complex C^*, d, fix a direct sum decompositions of vector spaces

$$C^q = B_q \oplus H_q \oplus A_q \tag{5.36 }$$

where $B_q = d^* C^{q+1}$ are the boundaries, $B_q \oplus H_q$ are the cocyles, and A_q fills out the rest of C_q. Note $A_0 = \{0\}$. Then by choice H_q is naturally isomorphic to the homology, $H_q \cong H_q(C^*, d^*)$ under the projection of Z_q onto $H_q(C^*, d^*)$.

Taking the wedge product in this order establishes a natural isomorphism

$$\Psi' : (det\ B_q) \otimes (det\ H_q) \otimes (det\ A_q) \cong det\ C^q \tag{5.37 }$$

Fix generators $c(q) \in det\ C^q, a'(q) \in det\ A_q$ for each $q = 0, \cdots, N$. Use the convention that for the 0-dimensional subspace, $\{0\}$, $det\ \{0\} = C$ by definition and the chosen generator is $1 \in C$.

Let $d^*(a'(q))$ denote the generator of $det\ B_{q-1}$ determined by the isomorphism $d^* : A_q \to B_{q-1}$. Then for each $q = 0, \cdots, N$ there is a unique element $h'(q) \in det\ H_q$ such that

$$c(q) = \Psi'(\ d^*(a'(q+1))\ \otimes\ h'(q)\ \otimes\ a'(q)\ ) \tag{5.38 }$$

With these choices, the algebraic torsion mapping

$$\tau' : det(C^*) \overset{\cong}{\to}\ det(H_*(C^*, d)) \tag{5.39 }$$

is defined as the isomorphism which maps

$$\begin{aligned} &c(0) \otimes c(1)^{-1} \otimes \cdots \otimes c(N)^{(-1)^N} \\ &\mapsto h'(0) \otimes h'(1)^{-1} \otimes \cdots \otimes h'(N)^{(-1)^N} \end{aligned}$$

Concretely, having already chosen bases $\mathbf{c}^q$ for C^q, let $\mathbf{b}_q = \{b_{q,1}, b_{q,2}, \cdots, b_{q,r_q}\}$ be an ordered basis for the boundaries $B_q = Im\ d^* : C^{q+1} \to C^q$ in C^q. Now chose $\tilde{\mathbf{b}}_q = \{\tilde{b}_{q,1}, \tilde{b}_{q,2}, \cdots, \tilde{b}_{q,r_q}\}$ in C^{q+1} so that $d^*\tilde{b}_{q,i} = b_{q,i}$ in B_q for $i = 1, \cdots, r_q$. Thus taking the ordered elements $\tilde{\mathbf{b}}_q$ as a basis for A_{q+1} and $\mathbf{b}_q$ as a ordered basis for B_q, one has $d^*(\wedge_j \tilde{b}_{q,j}) = (\wedge_j b_{q,j}) \in det\ B_q$ under the isomorphism $d^* : A_{q+1} \to B_q$.

Now $Z_q/B_q \cong H_q$, where Z_q are the q-cycles, $Z_q = ker\ d^* : C^q \to C^{q-1}$. Pick elements $lh_{q,k} \in Z_q \subset C^q$ projecting to the chosen ordered basis elements $h'_{q,k}$ of H_q. Taking the ordered set $\mathbf{lk}_q := \{lh_{q,k} | k = 1, \cdots, u_k\}$ with $u_q = dim_C\ H_q(C^*, d^*)$ as a basis for $H_q \subset C^q$ gives the direct sum decomposition, $C^q = B_q \oplus H_q \oplus A_{q-1}$, since with these choices $\{\mathbf{b}_q, \mathbf{lh}_q, \tilde{\mathbf{b}}_{q-1}\}$ is an ordered basis for C^q. This can be compared to the chosen basis $\mathbf{c}^q$.

For the above choices one sets

$$\tau'(C^*, d^*, \{\mathbf{c}^q\}, \{\mathbf{h'}_q\}) = \left[\prod_{q=0}^{N} [\mathbf{b}_q, \mathbf{lh}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}^q]^{(-1)^q} \right]^{-1} \tag{5.40 }$$

. A simple check shows it is independent of all choices but those of $\{\mathbf{c}_q\}, \{\mathbf{h}_q\}$.

The desired isomorphism τ' is then the element of $Hom(det(C^*), det(H_*(C^*, d^*)))$ defined by

$$\tau'(c(0) \otimes c(1)^{-1} \otimes \cdots \otimes c(N)^{(-1)^N})$$
$$= \tau'(C^*, d^*, \{\mathbf{c}^q\}, \{\mathbf{h'}_q\}) \times (h'(0) \otimes h'(1)^{-1} \otimes \cdots \otimes h'(N)^{(-1)^N})$$

In terms of the above notation, if (C^*, d, d^*) has the two differentials d, d^*, then for choices of ordered bases $\{\mathbf{c}^q\}$, $\{\mathbf{h}^q\}$, and $\{\mathbf{h}_q\}$ for C^q, $H^q(C^*, d)$, and $H_q(C^*, d^*)$, respectively, one may form the quotient

$$\tau(C^*, d, \{\mathbf{c}^q\}, \{\mathbf{h}^q\}) / \tau'(C^*, d^*, \{\mathbf{c}^q\}, \{\mathbf{h'}_q\}).$$

Consequently one has a natural invariant $torsion(C^*, d, d^*, \{\mathbf{h}^q\}, \{\mathbf{h'}_q\})$ defined in several equivalent manners:

$$torsion(C^*, d, d^*, \{\mathbf{h}^q\}, \{\mathbf{h'}_q\})$$
$$:= (-1)^{S(C^*)}\ \tau(C^*, d, \{\mathbf{c}^q\}, \{\mathbf{h}^q\}) / \tau'(C^*, d^*, \{\mathbf{c}^q\}, \{\mathbf{h}_q\})$$
$$= (-1)^{S(C^*)} \prod_q \left[\frac{[\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{c}^q]}{[\mathbf{b}_q, \mathbf{lh}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}^q]} \right]^{(-1)^{q+1}}$$
$$= (-1)^{S(C^*)} \prod_q [\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{b}_q, \mathbf{lh}_q, \tilde{\mathbf{b}}_{q-1}]^{(-1)^{q+1}}$$

The last makes the independence of the choice of basis for C^q clear, so the definition depends only on (C^*, d, d^*) and the choice of bases for the cohomology and homologies.

Here the sign $(-1)^{S(C^*)}$ is defined as in equation 5.28 .

As the above makes clear, the non-zero complex number $torsion(C^*, d, d^*, \{\mathbf{h}^q\}, \{\mathbf{h}_q\})$ determines the torsion $torsion(C^*, d, d^*)$ by the formula:

$$torsion(C^*, d, d^*)(\Lambda\{\mathbf{h}_q\})$$
$$= torsion(C^*, d, d^*, \{\mathbf{h}^q\}, \{\mathbf{h'}_q\}) \cdot (\Lambda\{\mathbf{h}_q\})$$

an element of $Hom(det(H_*(C^*, d^*), det(H^*(C^*, d)))$.

Properties of algebraic torsion, $torsion(C^*, d, d^*)$**:**

It will be helpful to clarify the relations between these approaches. This is accomplished by the following claims. The mysterious sign $(-1)^{S(C^*)}$ in 5.28 is chosen so these properties hold exactly.

Claim A: [*Relation to Bilinear pairings*] Let (C^*, d) be a finite cochain complex let $\{c^{q,j} | j = 1, \cdots, n_q\}$ be the specified basis for C^q and $\{\ h^{q,j} | j =$

$1\cdots,v_q\}$ be the specified basis for $H^q(C^*,d)$. Thus d is a differential of degree $+1$.

Introduce non-degenerate complex bilinear pairings

$$(.\ ,\ .):\ C^q\times C^q\to C$$

defined by $(c^{q,i},c^{q,j})=\delta_{i,j}$. That is, $(\Sigma_i\ \lambda_i\ c^{q,i},\Sigma_j\ \mu_j\ c^{q,j})=\Sigma_j\ \lambda_j\ \mu_j$. Note $(.,.)$ is not Hermitian, rather $(\lambda a,b)=\lambda(a,b)=(a,\lambda b)$ for a complex number λ.

Let d^* be unique differential of degree -1 with $(da,b)=(a,d^*b)$, i.e., the dual of d under $(.,.)$. Let $\mathbf{h'}_q=\{h'_{q,j}|j=1\cdots,v_q\}$ be the dual basis in $H_q(C^*,d^*)$ to the chosen basis $\{\ h^{q,j}|j=1\cdots,v_q\}$ in $H^q(C^*,d)$ under this pairing, $(.,.)$. It is claimed:

$$torsion(C^*,d,d^*,\{\mathbf{h}^q\},\{\mathbf{h'}_q\})=\tau(C^*,d,\{\mathbf{h}^q\},\{\mathbf{c}_q\})^2$$

The significance of this equation is that the square of the traditional torsion appears naturally in the bi-complex setting.

Claim B:[*Relation to Eigenvalues*] Suppose (C^*,d,d^*) is a finite bi-complex as above. Assume moreover that the combinatorial Laplacian, $\Delta:=(d\,d^*{+}d^*\,d)$, has no eigenvalue 0. Record the eigenvalues counted with multiplicities of $\Delta_q=(\Delta|C^q):C^q\to C^q$ as $\{\lambda_{q,i}|i=1,\cdots,n_q\}$. By assumption they are all non-zero.

It is claimed that the cohomologies and homologies $H_q(C^*,d)$ and $H^q(C^*,d^*)$ vanish, so no choice of basis is needed. Moreover, the algebraic torsion of (C^*,d,d^*), a complex number, is expressible as:

$$\begin{aligned}&torsion(C^*,d,d^*) \qquad\qquad (5.41\ )\\ &=[\prod_q\ (det(\Delta_q))^{(-1)^q}]^{-1}=[\prod_{q=0}^{N}\ (\prod_{j=1}^{n_q}\ \ \lambda_{q,j})^{(-1)^q q}]^{-1}\end{aligned}$$

This allows the right hand side to be extended from the setting of non-zero eigenvalues to the general case. This is precisely what is needed in the situation of the zero-mode correction.

Claim C:[*Stability Property*] Let (C^*,d,d^*) be a finite chain complex with differentials, d,d^* of degrees $+1,-1$ respectively.

Suppose specified bases of the cohomologies, $H^q(C^*,d)$, say $\mathbf{h}^q=\{h^{q,j}|j=1\cdots,v_q\}$ with $v_q=dim_C H^q(C^*,d)$, and there are also given specified bases of the homologies, $H_q(C^*,d^*)$, say $\mathbf{h}_q=\{h_{q,j}|j=1\cdots,u_q\}$ with $u_q=dim_C\ H_q(C^*,d^*)$.

Moreover, let (D^*,d',d'^*) be another finite chain complex with differentials d',d'^* of degrees $+1,-1$ respectively and suppose that the cohomology $H^*(D^*,d')$ and homology $H_*(D^*,d'^*)$ vanish. That is, D^* is acyclic under both d' and d'^*.

In this circumstance, one has the natural isomorphisms, $H^*(C^* \oplus D^*, (d \oplus d')) = H^*(C^*, d)$ and $H^*(C^* \oplus D^*, (d^* \oplus d'^*)) = H_*(C^*, d^*)$, so the choices above may be used in computing the torsion of the bi-complex, $(C^* \oplus D^*, d+d', d^*+d'^*)$. It is claimed:

$$\begin{aligned}&torsion(C^* \oplus D^*, (d \oplus d'), (d^* \oplus d'^*), \{\mathbf{h}^q\}, \{\mathbf{h}_q\})\\&= torsion(C^*, d, d^*, \{\mathbf{h}^q\}, \{\mathbf{h}_q\}) \cdot torsion(D^*, d', d'^*)\end{aligned}$$

Here by acyclicity, $torsion(D^*, d', d'^*)$ is just a complex number.

This stability property is particularly desirable, and depends on the choice of signs in 5.28 . In the case that $\Delta|D_*$ has all non-zero eigenvalues, the term $torsion'(D^*, d, d^*)$ is expressible by claim B in terms of those eigenvalues.

In view of these formulas, one deduces the equation 4.21 of section 3. This proves that the invariant defined in theorem 4.4 is independent of the choice of $K > 0$.

Details of Proofs of Claims A,B,C:

A: Here since $c^{q,i}, i = 1, \cdots, n_q$ is a basis for C^q, and $B^{q+1} = dC^q$ has rank s_{q+1}, one may reorder this basis so the last s_{q+1} elements, that is, $\{c^{q,n_q - s_{q+1}+1}, \cdots, c^{n_q}\}$ project by d to a basis of B^{q+1}. Set $b^{q+1,i} = dc^{q,i}$, for $n_q - s_{q+1} + 1 \le i \le n_q$, giving a basis $\mathbf{b}^{q+1}$ for B^{q+1}, and set $\tilde{b}^{q+1,i} = c^{q,i}$. Let $\tilde{\mathbf{b}}^{q+1} = \{\tilde{b}^{q+1,i}\} = \{c^{q,i} | n_q - s_{q+1} + 1 \le i \le n_q\} \subset C^q$. The span of these last elements maps bijectively to $B^{q+1} = dC^q$.

Now for each i, $1 \le i \le n_q - s_{q+1}$, there are unique numbers $U_{q,i,j}$, $n_q - s_{q+1} + 1 \le j \le n_q$ with $dc^{q,i} = \Sigma_{j=n_q-s_{q+1}+1}^{n_q} U_{q,i,j}\ dc^{q,j}$. Thus, $z^{q,i} = c^{q,i} - \Sigma_{j=n_q-s_{q+1}+1}^{n_q} U_{q,i,j}\ c^{q,j}$, for $1 \le i \le n_q - s_{q+1}$, provides a basis for the cycles $Z^q = (ker\ d : C^q \to C^{q+1})$. Since Z^q projects onto $H^q(C^*, d)$ which is of dimension v_q, we can and do reorder these $n_q - s_{q+1}$ entries so that the v_q elements $\{z^{q,i} | n_q - s_{q+1} - v_q + 1 \le i \le n_q - s_{q+1}\}$ project to a basis of $H^q(C^*, d)$.

Since $C^q/Z^q \cong B^{q+1}$ under d, one has $n_q - (s_q + v_q) = s_{q+1}$ or equivalently $n_q = s_q + s_{q+1} + v_q$.

Set $lh^{q,k} = z^{q,n_q - s_{q+1} - v_q + k} = z^{q,s_q+k}$ for $1 \le k \le v_q$, and set $\mathbf{lh}^q = \{z^{q,s_q+k} | 1 \le k \le v_q\} = \{c^{q,s_q+k} - \Sigma_{j=n_q-s_{q+1}+1}^{n_q} U_{q,s_q+k,j}\ c^{q,j} | 1 \le k \le v_q\}$. These q-cocycles project to a basis for $H^q(C^*, d)$.

With these choices one has the ordered basis

$$\begin{aligned}&\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}\\&= \{dc^{q-1,i} | n_{q-1} - s_q + 1 \le i \le n_{q-1}\}\\&\qquad \sqcup \{c^{q,s_q+k} - \Sigma_{j=n_q-s_{q+1}+1}^{n_q} U_{q,s_q+k,j}\ c^{q,j} | 1 \le k \le v_q\}\\&\qquad \sqcup \{c^{q,j} | n_q - s_{q+1} + 1 \le j \le n_q\}\end{aligned}$$

of C^q to compare to the basis $\{c^{q,i} | 1 \le i \le n_q\}$. Here $n_q - s_{q+1} = s_q + v_q$.

If $dc^{q-1,i} = \Sigma_{j=1}^{s_q} R_{q,i,j}\ c^{q,j} + \Sigma_{j=s_q+1}^{s_q+v_q} S_{q,i,j}\ c^{q,j} + \Sigma_{j=s_q+v_q+1}^{n_q} T_{q,i,j}\ c^{q,j}$, for $n_{q-1} - s_q + 1 \le i \le n_{q-1}$. Then the change of basis matrix from $\mathbf{c}^q$ to

$\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}$ is

$$M_q = \begin{pmatrix} R_q & 0 & 0 \\ S_q & I & 0 \\ T_q & -U_q & I \end{pmatrix}$$

In particular, $det(R_q) \neq 0$ and for the above explicit bases $\mathbf{h}^q = \{[z^{q,j}] | s_q + 1 \leq j \leq s_q + v_q\}$ for $H^q(C^*, d)$

$$\begin{aligned} &\tau(C^*, d, \{\mathbf{h}^q\}, \{\mathbf{c}^q\}) := \prod_{q=0}^N \ [\mathbf{b}^q, \mathbf{lh}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{c}^q]^{(-1)^{q+1}} \\ &= \prod_q \ (det(R_q))^{(-1)^{q+1}} \end{aligned}$$

Since reordering the basis elements of $\{\mathbf{c}_q\}$ leaves the square unchanged, one may use this specific choice of ordering to compute the square $\tau(C^*, d, \{\mathbf{h}^q\}, \{\mathbf{c}^q\})^2 = (\prod_q \ (det(R_q)^{(-1)^{q+1}})^2$.

Now consider elements $z^*_{q,i} \in C^q$ for $1 \leq i \leq v_q$ of the special form $= c^{q,s_q+i} + \Sigma_{j=1}^{s_q} V_{q,i,j} \ c^{q,j}$. It is claimed that there is a unique choice of the $V_{q,i,j}$'s which makes these elements lie in the q-cycles for d^*, the dual of d under the pairing $(.,.)$. Here the q-cycles are $Z_q = (ker \ d^* | C^q : C^q \to C^{q-1})$.

The non-degenerate bilinear pairing $(.,.)$ on C_q is completely specified by $(c^{q,i}, c^{q,j}) = \delta_{i,j}$, the Kronecker delta function. This induces the identification of C^q with its dual, $(C^q)^* = Hom(C^q, C)$, given by the specified choices of bases.

The constraint $z^*_{q,i} \in Z_q$ is the same as $(z^*_{q,i}, B^q) = 0$. Since $\{d \, c^{q-1,n_{q-1}-s_q+k} | 1 \leq k \leq s_q\}$ is a basis for B^q, these constraints are the same as the s_q equalities: $0 = (z^*_{q,i}, d \ c^{q-1,n_{q-1}-s_q+k}) = (c^{q,s_q+i} + \Sigma_{j=1}^{s_q} \ V_{q,i,j} \ c^{q,j}, \Sigma_{m=1}^{s_q} \ R_{q,k,m} \ c^{q,m} + \Sigma_{m=s_q+1}^{s_q+v_q} \ S_{q,k,m} \ c^{q,m} + \Sigma_{m=s_q+v_q+1}^{n_q} \ T_{q,k,m} \ c^{q,m}) = (\Sigma_{j=1}^{s_q} \ V_{q,i,j} \ c^{q,j}, \Sigma_{m=1}^{s_q} \ R_{q,k,m} \ c^{q,m}) + (c^{q,s_q+i}, \Sigma_{m=s_q+1}^{s_q+v_q} \ S_{q,k,m} \ c^{q,m}) = \Sigma_{j=1}^{s_q} \ V_{q,i,j} \ R_{q,k,j} + S_{q,k,s_q+i} = (V \cdot R^t)_{i,k} + (S^t)_{s_q+i,k}$. Since R is invertible, the equations $(V \cdot R^t)_{i,k} = -(S^t)_{s_q+i,k}$ determine V uniquely. The claim follows.

Here the rank of $d : C^q \to C^{q+1}$ is the same as the rank of the adjoint via $(.,.)$ which is $d^* : C^q \to C^{q+1}$. That is, $s_{q+1} = dim \ B^{q+1} = dim \ B_q$.

Let $lh'_{q,i} = c^{q,s_q+i} + \Sigma_{j=1}^{s_q} \ V_{q,i,j} \ c^{q,j}$ for $1 \leq i \leq v_q$ for these unique choices. Then $lh'_{q,i} \in Z_q$ represent elements of $H_q(C^*, d^*)$ and moreover the pairing with the cycle representatives $lh^{q,j}$ specifying a basis for $H^q(C^*, d)$ is: $(lh'_{q,i}, lh^{q,j}) = (c^{q,s_q+i} + \Sigma_{j=1}^{s_q} \ V_{q,i,j} \ c^{q,j}, c^{q,s_q+k} - \Sigma_{j=n_q-s_{q+1}+1}^{n_q} \ U_{q,s_q+k,j} \ c^{q,j}) = \delta_{i,j}$.

Hence, the cycles $lk'_{q,i}$ represent in $H_q(C^*, d^*)$ the dual basis to the basis represented by the cocycles $lh^{q,j}$ in $H^q(C^*, d)$. Let $\mathbf{lh}'_q = \{lh'_{q,i} | 1 \leq i \leq v_q\}$.

Since R has non-zero determinant, and $\{dc^{q,n_q-s_{q+1}+k} | 1 \leq k \leq s_{q+1}\}$ form a basis for B^{q+1}, the elements $\{d^* c^{q+1,j} | 1 \leq j \leq s_{q+1}\}$ form a basis for $B_q = d^* C^{q+1}$. So one may take $\mathbf{b'}_q = \{d^* c^{q+1,j} | 1 \leq j \leq s_q\}$ as a basis for B_q and set $\tilde{\mathbf{b'}}_q = \{c^{q+1,j} | 1 \leq j \leq s_{q+1}\}$.

Indeed, for $1 \leq j \leq s_{q+1}, 1 \leq k \leq s_{q+1}$, one computes $(d^* c^{q+1,j}, c^{q,n_q-s_{q+1}+k})$ $= (c^{q+1,j}, dc^{q,n_q-s_{q+1}+k}) = (c^{q+1,j}, \Sigma_{m=1}^{s_{q+1}} \ R_{q+1,k,m} \ c^{q+1,m}) = R_{q+1,k,j} =$

$(R_{q+1,k,j}\ c^{q,n_q-s_{q+1}+k}, c^{q,n_q-s_{q+1}+k})$, so $d^* c^{q+1,j} = \Sigma_{m=1}^{s_{q+1}} R_{q+1,m,j}\ c^{q,s_q+v_q+m} + \Sigma_{m=1}^{s_q} X_{q,k,m}\ c^{q,m} + \Sigma_{m=s_q+1}^{s_q+v_q} Y_{q,k,m}\ c^{q,m}$ for some constants $X_{q,i,j}, Y_{q,i,j}$.

Consequently, one gets an ordered basis

$$\begin{aligned}&\mathbf{b}'_q, \mathbf{lh}'_q, \tilde{\mathbf{b}'}_{q-1} \\ &= \{d^* c^{q+1,i} | 1 \leq i \leq s_{q+1}\} \sqcup \{c^{q,s_q+i} + \Sigma_{j=1}^{s_q} V_{q,i,j}\ c^{q,j} | 1 \leq i \leq v_q\} \\ &\qquad \sqcup \{c^{q,i} | 1 \leq i \leq s_q\}\end{aligned}$$

for C^q. The change of basis matrix to $\mathbf{c}^q$ is:

$$N_q = \begin{pmatrix} X & V & I \\ Y & I & 0 \\ (R_{q+1})^t & 0 & 0 \end{pmatrix}$$

Hence, one sees that for this ordering of the basis $\{c^{q,i}\}$,

$$\begin{aligned}&\tau'(C^*, d^*, \{\mathbf{h}_q\}, \{\mathbf{c}_q\}) = \prod_q\ (det(N_q)^{(-1)^{q+1}} \\ &= \ \prod_q\ ((-1)^{s_{q+1}s_q + s_q v_q + s_{q+1} v_q} det(R_{q+1}))^{(-1)^{q+1}} \\ &= (-1)^R\ \prod_q\ [(det(R_q))^{(-1)^{q+1}}]^{-1}\end{aligned}$$

with the sign $(-1)^R = (-1)^{s_{q+1}s_q + s_q v_q + s_{q+1} v_q}$ coming from permuting the blocks $I, I, (R_{q+1})^t$ which have ranks s_{q+1}, s_q, v_q respectively.

Recall that $s_{q+1} = dim\ B^{q+1} = dim\ B_q$ and $v_q = dim\ H^q = dim\ H_q$.

But by equation 5.28 , $S(C^*) = \Sigma_q\ S_q$ with $S_q = [dim\ B_{q-1}(C^*) \cdot dim_C\ B^{q+1}(C^*) + dim_C\ B^{q+1}(C^*) \cdot dim_C H_q(C^*, d) + dim_C\ B_{q-1}(C^*) \cdot dim_C H^q(C^*, d^*)]$. So this special case, $S_q = s_q s_{q+1} + s_{q+1} v_q + s_q v_q$. Consequently, $(-1)^R = (-1)^{S(C^*)}$ with $S(C^*)$ as in 5.28

Hence,

$$\begin{aligned}&torsion(C^*, d, d^*, \{\mathbf{h}^q\}, \{\mathbf{h}'_q\}) \\ &= (-1)^{S(C^*)}\ \prod_{q=0}^N\ [\mathbf{b}^q, \mathbf{h}^q, \tilde{\mathbf{b}}^{q+1} / \mathbf{b}'_q, \mathbf{h}'_q, \tilde{\mathbf{b}'}_{q-1}]^{(-1)^{q+1}} \\ &= (-1)^{S(C^*)}\ [\prod_q\ (det(R_q))^{(-1)^{q+1}}] / [(-1)^R [\prod_q\ (det(R_q))^{(-1)^{q+1}}]^{-1}] \\ &= \ [\prod_q\ (det(R_q))^{(-1)^{q+1}}]^2 \\ &= \ [\tau(C^*, d, \{\mathbf{h}_q\}, \{\mathbf{c}^q\})]^2\end{aligned}$$

This proves claim A for a special choice of bases $\mathbf{h}_q$. A change of basis of $H_q(C^*, d)$ modifies both sides by the same non-zero multiple so it suffices to prove this result for any specific choice. Changing the ordering of the $\mathbf{c_q}$ also leaves both sides unchanged, so claim A is established.

B: If Δ has no vanishing eigenvalues, then Δ is an isomorphism, so in particular, for each $x \in C^q$ there is a $y \in C^q$ with $x = \Delta y = (d^* dy + dd^* y)$. Hence, $C^q = d^* C^{q+1} + dC^{q-1}$. On the other hand, if $x \in d^* C^{q+1} \cap dC^{q-1}$, then

$d^*x = 0, dx = 0$ by $(d^*)^2 = 0, (d)^2 = 0$, so one gets $\Delta x = (d^*d + dd^*)x = 0$ also which implies $x = 0$. Hence, one has a direct sum splitting of C^q as:

$$C^q = d^*C^{q+1} \oplus dC^{q-1}$$

By $d^*\Delta = d^*dd^* = \Delta d^*, d\Delta = dd^*d = \Delta d$ the isomorphism Δ preserves the above splitting. Hence, Δ maps d^*C^{q+1} and dC^{q-1} isomorphically to themselves. Since $\Delta|d^*C^* = d^*d$ and $\Delta|dC^* = dd^*$, these maps are isomorphisms on d^*C^{q+1}, dC^{q-1} respectively. In particular, $d : d^*C^{q+1} \to dC^q$ is an injection and moreover $d^* : dC^q \to d^*C^{q+1}$ is an injection with composite an isomorphism. Hence, one finds all these maps are necessarily isomorphisms. In summary, one has induced isomorphisms:

$$d^*C^{q+1} \xrightarrow{d} dC^q \quad and \quad dC^q \xrightarrow{d^*} d^*C^{q+1} .$$

In particular, $H_*(C^*, d) = 0$ and $H^*(C^*, d^*) = 0$, as claimed. Also, by $H^*(C^*, d) = 0$, $n_q = s_q + s_{q+1}$.

Now take a basis, say $dy_{q,i}, 1 \leq i \leq s_q$ for $B^q = dC^{q-1}$ for which the matrix Δ has matrix representative in Jordan block form. Then

$$\Delta\ dy_{q,i} = dd^*\ dy_{q,i} = \lambda_{q,i}\ dy_{q,i} + \Sigma_{j<i}\ T_{q,ij}\ dy_{q,j} \quad 1 \leq i \leq s_q$$

for some constants T_{ij}. Here the $\lambda_{q,i} \neq 0$ are the generalized eigenvalues of $\Delta|dC^{q-1}$ counted with their multiplicities. Also $y_{q,i} \in C^{q-1}$.

Since $\lambda_{q,i} \neq 0$ for any i, one may invert the above to get unique constants $U_{q,i,j}$ with

$$dd^*(dy_{q,i} + \Sigma_{j<i}\ U_{q,i,j}\ dy_{q,j}) = \lambda_{q,i}\ dy_{q,i}$$

Let

$$x_{q,i} = (1/\lambda_{q,i})(dy_{q,i} + \Sigma_{j<i}\ U_{q,i,j}\ dy_{q,j}) \quad so \quad dd^*\ x_{q,i} = dy_{q,i}$$

with $x_{q,i} \in C^q$.

Since the $\{dy_{q,i}|1 \leq i \leq s_q\}$ are a basis for $B^q = dC^{q-1}$ and dd^* is an isomorphism of dC^{q-1} onto itself, then necessarily $x_{q,i}$ is also a basis for $B^q = dC^{q-1}$. By $d^*|dC^{q-1}$ an isomorphism to d^*C^q, it follows that $\{d^*x_{q,i}|1 \leq i \leq s_q\}$ is a basis for d^*C^q. In particular, in view of $C^q = dC^{q-1} \oplus d^*C^{q+1}$ one obtains that

$$\mathbf{c}_q = \{dy_{q,j}|1 \leq j \leq s_q\} \sqcup \{d^*\ x_{q+1,i}|1 \leq i \leq s_{q+1}\}$$

is a basis for C^q. In particular,

$$dim\ B_q = dim\ B^{q+1} = s_{q+1} \tag{5.42 }$$

The computation of the torsion $\tau(C^*, d, \{\mathbf{c}^q\})$ proceeds as follows for these choices.

For this choice of ordered bases $\mathbf{c}^q$, one chooses $\mathbf{b}^q$, the ordered elements $b^{q,i} = dy_{q,i}$, as basis for $B^q = dC^{q-1}$ inside C^q. One specifies $\tilde{\mathbf{b}}^q$ by $\tilde{b}_{q,i} = d^*x_{q+1,i}$ since $dd^*x_{q+1,i} = dy_{q+1,i}$. With these choices

$$[\mathbf{b}^q, \tilde{\mathbf{b}}^{q+1}/\mathbf{c}^q] = [\{dy_{q,j}\} \sqcup \{d^*x_{q+1,i}\}/\{dy_{q,j}\} \sqcup \{d^*x_{q+1,i}\}] = +1$$

Hence, $\tau(C^*, d, \{\mathbf{c}^q\}) = +1$ for these choices.

In order to compute $\tau'(C^*, d^*, \{\mathbf{c}^q\})$ one may take $\mathbf{b'}_q$ to be the basis $\{d^* x_{q+1,i} | 1 \leq i \leq s_{q+1}\}$ of $B_q = d^* C^{q+1}$ and $\tilde{\mathbf{b'}}_q$ to be the elements $\{x_{q,j} | 1 \leq j \leq s_q\}$. Hence,

$$[\mathbf{b'}_q, \tilde{\mathbf{b'}}_{q-1}/\mathbf{c}_q] = [\{d^* x_{q+1,i}\} \sqcup \{x_{q,j}\}/\{dy_{q,j}\} \sqcup \{d^* x_{q+1,i}\}]$$

In view of the equalities $x_{q,i} = (1/\lambda_{q,i})(dy_{q,i} + \Sigma_{j<i} U_{q,ij}\ dy_{q,j})$, this implies that

$$[\mathbf{b'}_q, \tilde{\mathbf{b'}}_{q-1}/\mathbf{c}_q] = (-1)^{s_q s_{q+1}} (\prod_i (\lambda_{q,i})^{-1})$$

since $\flat\{dy_{q,j}\} = s_q$ and $\flat\{d^* x_{q+1,i}\} = s_{q+1}$

In toto one gets:

$$\begin{aligned} &torsion(C^*, d^*, d) \\ &= (-1)^{S(C^*)}\ \tau(C^*, d, \{\mathbf{c}_q\})/\tau(C^*, d^*, \{\mathbf{c}_q\}) \\ &= (-1)^{S(C^*)}\ (-1)^R \prod_q (\prod_i \lambda_{q,i})^{(-1)^{q+1}} \end{aligned}$$

with $R = \Sigma_q\ s_q s_{q+1}$.

Here by equation 5.28 , $S(C^*) = \Sigma_q\ S_q$ with $S_q = [dim\ B_{q-1}(C^*) \cdot dim_C\ B^{q+1}(C^*) + dim_C\ B^{q+1}(C^*) \cdot dim_C H_q(C^*, d) + dim_C\ B_{q-1}(C^*) \cdot dim_C H^q(C^*, d^*)]$.

In this doubly acyclic case $S_q = (dim\ B_{q-1})(dim B^{q+1}) = s_q s_{q+1}$ via the equality $s_q = (dim\ B_{q-1})$ observed in equation 5.42 . Hence, $(-1)^R = (-1)^{S(C^*)}$ and so

$$torsion(C^*, d^*, d) = \prod_q (\prod_i \lambda_{q,i})^{(-1)^{q+1}} \tag{5.43 }$$

But the eigenvalues of Δ_q are those of $\Delta | dC^{q-1}$, i.e., $\{\lambda_{q,i} | 1 \leq i \leq s_q\}$ and those of $\Delta | d^* C^{q+1}$. These last under the isomorphism d^* from dC^q to $d^* C^{q+1}$ are the eigenvalues $\{\lambda_{q+1,j} | 1 \leq j \leq s_{q+1}\}$.

That is, $det(\Delta_q) = (\prod_i\ \lambda_{q,i})(\prod_j\ \lambda_{q+1,j})$. In particular, as in Ray and Singer [30]

$$\prod_q (det(\Delta_q))^{(-1)^q q} = \prod_{q>0} (\prod_i \lambda_{q,i})^{(-1)^q}.$$

Substituting this into the above gives:

$$torsion(C^*, d, d^*) = [\prod_q (det(\Delta_q))^{(-1)^q q}]^{-1}$$

as claimed.

C: Make choices for (C^*, d, d^*) and make corresponding choices for (D^*, d', d'^*). Comparing terms one finds the same basis entries occurring except for the ordering of the terms. For example, in $C^* \oplus D^*$ the determinants

$$[\mathbf{b}(C^*)^q \sqcup \mathbf{b}(D^*)^q, \mathbf{h}(C^*)^q, \tilde{\mathbf{b}}(C^*)^{q+1} \sqcup \tilde{\mathbf{b}}(D^*)^{q+1} \\ /\mathbf{b}(C^*)_q \sqcup \mathbf{b}(D^*)_q, \mathbf{h}(C^*)_q, \tilde{\mathbf{b}}(C^*)_{q-1} \sqcup \tilde{\mathbf{b}}(D^*)_{q-1}]$$

with the evident notation occurs, while the corresponding term for C^* is $[\mathbf{b}(C^*)^q, \mathbf{h}(C^*)^q, \tilde{\mathbf{b}}(C^*)^{q+1}/\mathbf{b}(C^*)_q, \mathbf{h}(C^*)_q, \tilde{\mathbf{b}}(C^*)_{q-1}]$ and that for D_* is $[\mathbf{b}(D^*)^q, \tilde{\mathbf{b}}(D^*)^{q+1}/\mathbf{b}(D^*)_q, \tilde{\mathbf{b}}(D^*)_{q-1}]$. The first of these is the product of the next two times a sign $(-1)^{T_q}$ which accounts for the change of ordering.

Let $r_q = dim_C\ B_q(C^*) = dim\ im\ d^*_{q+1}[C^{q+1}],\quad s_q = dim_C\ B^q(C^*) = dim\ im\ d_{q-1}\ [C^{q-1}]$, $u_q = dim_C\ H_q(C^*, d^*), v_q = dim_C\ H^q(C^*, d)$. Let $x_q = dim_C\ B_q(D^*) = dim\ im\ d^*_{q+1}[D^{q+1}]$, $y_q = dim_C\ B^q(D^*) = dim\ im\ d_{q-1}[D^{q-1}]$. With the above notation, the number of elements in $\mathbf{b}(D^*)_q$ is x_q and each must be passed by $\mathbf{h}(C^*)_q, \tilde{\mathbf{b}}(C^*)_{q-1}$ which have $u_q + r_{q-1}$ elements to get the desired order. Similarly, the number of elements in $\mathbf{b}(D_*)^q$ is y_q and each must be passed by $\mathbf{h}(\mathbf{C}^*)^{\mathbf{q}}, \tilde{\mathbf{b}}(\mathbf{C}^*)^{\mathbf{q+1}}$ which has $v_q + s_{q+1}$ to get the correct order. Consequently, multiplying over q gives the relation

$$\begin{aligned} &torsion'(C^* \oplus D^*, (d + d'), (d^* + d'^*), \{\mathbf{h}^q\}, \{\mathbf{h}_q\}) \\ &= (-1)^{S(C^* \oplus D^*)}(-1)^{S(C^*)}(-1)^{S(D^*)}(-1)^T \\ &\quad \cdot torsion'(C^*, d, d^*, \{\mathbf{h}_q\}, \{\mathbf{h}_q\}) \cdot torsion'(D_*, d, d^*) \end{aligned}$$

where by the above $T = \Sigma_q\ T_q$ with $T_q = x_q(u_q + r_{q-1}) + y_q(v_q + s_{q+1})$.

Now by $B_q \cong C_{q+1}/Z_{q+1}, H_q \cong Z_q/B_q$ one has for $n_q = dim_C\ C^q$, the equations $r_0 = n_0 - u_0, r_1 = n_1 - u_1 - r_0, r_2 = n_2 - u_2 - r_1, \cdots$, which inductively gives

$$r_q = dim_C\ B^q(C^*) = (\Sigma^q_{i=0}\ (-1)^i\ n_{q-i}) - (\Sigma^q_{i=0}\ (-1)^i\ u_{q-i}). \qquad (5.44\ )$$

Similarly from $B^q \cong C_{q-1}/Z^{q-1}, H^q \cong Z^q/B^q$, one has the equations $s_1 = n_0 - v_0, s_2 = n_1 - v_1 - s_1, \cdots$ which inductively give

$$s_{q+1} = dim_C\ B^{q+1}(C^*) = (\Sigma^q_{i=0}\ (-1)^i\ n_{q-i}) - (\Sigma^q_{i=0}(-1)^i\ v_{q-i}). \qquad (5.45\ )$$

In particular, one has

$$s_{q+1} = r_q + \Sigma^q_{i=0}\ (-1)^i\ [u_{q-i} - v_{q-i}]. \qquad (5.46\ )$$

Applied to the doubly acyclic complex, (D^*, d, d^*), with $\hat{n}_q = dim_C\ D^q$, this yields the formulas:

$$y_{q+1} = x_q = \Sigma^q_{i=1}\ (-1)^i\ \hat{n}_{q-i} \qquad (5.47\ )$$

As an example of the usefulness of these relations, consider the difference

$$\begin{aligned} &D_1 := dim\ B_{q-1}(C^* \oplus D^*) \cdot dim_C\ B^{q+1}(C^* \otimes D^*) \\ &-dim\ B_{q-1}(C^*) \cdot dim_C\ B^{q+1}(C^*) - dim\ B_{q-1}(D^*) \cdot dim_C\ B^{q+1}(D^*) \end{aligned}$$

which in the above notation equals $r_{q-1}y_{q+1} + x_{q-1}s_{q+1}$. By equation 5.47 the last is precisely $r_{q-1}x_q + y_q s_{q+1}$ which is two of the terms in T_q.

The difference

$$\begin{aligned}&D_2 := dim_C\ B^{q+1}(C^* \oplus D^*) \cdot dim_C H_q(C^* \oplus D^*)\\ &-B^{q+1}(C^*) \cdot dim_C H_q(C^*, d) - B^{q+1}(D^*) \cdot dim_C H_q(D^*)\end{aligned}$$

via $H_q(D^*, d') = 0$ is in the above notation $B^{q+1}(D^*) \cdot u_q = y_{q+1}u_q = x_q u_q$ which is another of the terms in T_q.

Finally, the difference

$$\begin{aligned}&D_3 := B_{q-1}(C^* \oplus D^*) \cdot dim_C H^q(C^* \oplus D^*)\\ &-B_{q-1}(C^*) \cdot dim_C H^q(C^*, d^*) - B_{q-1}(D^*) \cdot dim_C H^q(D^*) \text{ via } H^q(D^*, d'^*) = 0\end{aligned}$$

is in the above notation $B_{q-1}(D^*) \cdot v_q = x_{q-1}v_q = y_q v_q$ which is the last term in T_q.

These last three formulas show that the sign $(-1)^{S(C^* \oplus D^*)}(-1)^{S(C^*)}(-1)^{S(D^*)}(-1)^T$ is always $+1$ proving claim C since $S(C^*)$ is defined precisely by $S(C^*) = \Sigma_q\ [dim\ B_{q-1}(C^*) \cdot dim_C\ B^{q+1}(C^*)$ $+ dim_C\ B^{q+1}(C^*) \cdot dim_C H_q(C^*, d^*) + dim_C\ B_{q-1}(C^*) \cdot dim_C H^q(C^*, d)]$.

6 Variation of algebraic torsion with changing Hermitian inner product on TW:

Let $g(u), a \leq u \leq b$, be a smoothly varying Hermitian inner product on the bundle TW. Suppose that $K > 0$ is not a generalized eigenvalue for any of the operators $\Delta_{E,\bar\partial,g(u)}$ associated to these Hermitian metrics, $g(u)$ acting on $A^{p,q}(W, E)$.

The spectral projections $\Pi_{E,K,g(u)}$ onto the finite dimensional subspaces $C^{p,*}(t) = \bigoplus_{q=0}^{N} C^{p,q}(E, K, g(u))$, the spans of the generalized eigenvalues with real part less than K, by standard spectral theory, are smoothly varying operators in u, for $a \leq u \leq b$.

Hence, for any fixed $u_0, a \leq u_0 \leq b$, the fixed spectral projection $\Pi_{E,K,g(u_0)}$ applied to $C^{p,q}(E, K, g(u))$ must be an isomorphism onto $C^{p,q}(E, K, g(u_0))$ for u sufficiently close to u_0. Pick an $\epsilon > 0$ so this is true for all p, q for all $|u - u_0| \leq \epsilon$ with $a \leq u \leq b$.

Let $P_{p,q}(u_0)$ be the kernel of the reference spectral projection $\Pi_{p,q,E,K}(u_0)$. Then one has a direct sum decomposition of $(L^{p,q})^2(W, E)$:

$$(L^{p,q})^2(W, E) = C^{p,q}(E, K, g(t_0)) \ \oplus\ P_{p,q}(t_0)$$

with the first summand of finite dimension and the second closed in $(L^{p,q})^2(W, E)$.

Recall that for two ordered bases, say $Y = \{y_1, \cdots, y_m\}$ and $X = \{x_1, \cdots, x_m\}$ of an m dimensional vector space V, the determinant of change of basis matrix $[Y/X]$ is the complex number with

$$y_1 \wedge \cdots \wedge y_m = [Y/X]\ x_1 \wedge \cdots \wedge x_m$$

If V is a subspace $j : V \subset W$ of a complex vector space, W, which is a direct sum $W = U_1 \oplus U_2$ and the projection, $\Pi : V \to U_1$, of W onto U_1 along U_2 is defines an isomorphism $V \cong U_1$, then one may compute the number $[Y/X]$ as follows:

Write out the elements, x_j, y_j in this decomposition, $x_j = \Pi(x_j) + (Id - \Pi)x_j, y_k = \Pi(y_j) + (Id - \Pi)y_k$ and expand the terms of the wedge product with respect to the direct sum decomposition

$$\wedge^m W \cong \oplus_{k=0}^m \wedge^k U_1 \otimes \wedge^{m-k} U_2$$

where $m = dim_C\ U_1$.

Comparing the entries in the summand $\bigwedge^m U_1$ gives the formula:

$$\Pi(y_1) \wedge \cdots \wedge \Pi(y_m) = [Y/X]\ \Pi(x_1) \wedge \cdots \wedge \Pi(x_m)$$

this computation being carried out in U_1. This method will be utilized for the above fixed decomposition at $u = u_0$.

Now to compute the change of algebraic torsion for varying u, for the family of complexes $(C^{p,*}(u), d, d^*) = (\oplus_q C^{p,q}(E, K, g(u)), d, d^*)$, one may first specify choices for $u = u_0$ and then for u nearby.

Choose an ordered basis, say $\mathbf{h}^q = \{h^{q,j} | j = 1 \cdots v_q\}$ for $H^{p,q}_{\bar{\partial}_E}(W, E)$. As noted above,

$$\begin{aligned} &H^q(C^{p,*}(u_0), d) = H^q(\oplus_{q'} C^{p,q'}(E, K, g(t_0), \bar{\partial}_E) \\ &\cong H^q(A^{p,*}(W, E), \bar{\partial}_E) \cong H^{p,q}_{\bar{\partial}_E}(W, E) \end{aligned}$$

so one may chose $lh^{q,j}(u_0) \in\ ker\ \bar{\partial}_E : C^{p,q}(E, K, g(u_0)) \to C^{p,q+1}(E, K, g(u_0))$ which project to the ordered basis $\{h^{q,j} | 1 \leq j \leq v_q\}$ chosen.

Let $\mathbf{lh}^q(u_0)$ be the ordered set $\{lh^{q,j}(u_0) | j = 1, \cdots v_q\}$.

Since $\hat{\star}_{g(u)} \otimes Id_E$ commutes with $\Delta_{E,\bar{\partial},g(u)}$, it maps $C^{p,q}(E, K, g(u))$ isomorphically to $C^{N-q,N-p}(E, K, g(u))$ and since $\hat{\star}_{g(u)}\ \hat{\star}_{g(u)} = \pm Id$, it interpolates the coboundary map D'' up to sign with the differential $d^* = -(\hat{\star}_{g(u)} \otimes Id_E)D''(\hat{\star}_{g(u)} \otimes Id_E)$. Hence, it suffices to make choices for the complex $(C^{N-*,N-p}(E, K, g(u)), D'')$ and transplant them via $\hat{\star}_{g(u)}$ to the complex $(C^{p,\star}(E, K, g(u)), d^*)$.

In a parallel fashion, choose an ordered basis, say $\mathbf{h'}^q = \{h'^{q,j} | j = 1 \cdots u_q\}$ for $H^{q,N-p}_{D''}(W, E)$. Again, $H^q(\oplus_{q'} C^{q',N-p}(E, K, g(u_0), D'') \cong H^{q,N-p}_{D''}(W, E)$ so one may chose $lh'^{q,j}(u_0) \in\ ker\ D'' : C^{q,N-p}(E, K, g(u_0)) \to C^{q+1,N-p}(E, K, g(u_0))$ which project to the ordered basis $\{h'^{q,j} | 1 \leq j \leq u_q\}$ chosen. Let $\mathbf{lh'}^q(u_0)$ be the ordered set $\{lh'^{q,j}(u_0) | j = 1, \cdots u_q\}$.

Next pick for each q an an ordered basis, say $\mathbf{b}^q(u_0) = \{b^{q,1}(u_0), \cdots, b^{q,s_q}(u_0)\}$ for the coboundaries, $B^q(u_0) = image\ \bar{\partial}_E : C^{p,q-1}(E, K, g(u_0)) \to C^{p,q}(E, K, g(u_0)$. Finally pick for each $b^{q,j}(u_0)$ an element $\tilde{b}^{q,j}(u_0) \in C^{p,q-1}(E, K, g(u_0))$ with $\bar{\partial}_E\ \tilde{b}^{q,j}(u_0) = b^{q,j}(u_0)$. Let $\tilde{\mathbf{b}}^q(u_0) =$

$\{\tilde{b}^{q,1}(u_0), \cdots, \tilde{b}^{q,s_q}(t_0)\}$ be ordered by increasing last index. Here $\tilde{\mathbf{b}}^q(u_0) \subseteq C^{p,q-1}(E, K, g(u_0))$.

In a parallel fashion, pick for each q an ordered basis, say $\mathbf{b'}^q(u_0) = \{b'^{q,j}(u_0), \cdots, b'^{q,s_q}(u_0)\}$ for the coboundaries for D'', *image* $D'' : | C^{q-1,N-p}(E, K, g(u_0)) \to C^{q,N-p}(E, K, g(u_0)$. Finally pick for each $b'^{q,j}(u_0)$ an element $\tilde{b'}^{q,j}(u_0) \in C^{q-1,N-p}(E, K, g(u_0))$ with $D''\ \tilde{b'}^{q,j}(u_0) = b'^{q,j}(u_0)$. Let $\tilde{\mathbf{b'}}^q(u_0) = \{\tilde{b'}^{q,1}(u_0), \cdots, \tilde{b'}^{q,s_q}(u_0)\}$ be ordered by increasing last index. Here $\{\tilde{\mathbf{b'}}^q(u_0)\} \subseteq C^{q,N-p}(E, K, g(u_0))$.

Since $\Delta_{E,\bar{\partial},g(u_0)}$ commutes with the differential $\bar{\partial}_E$ the spectral projection $\Pi_{E,K,g(u_0)}$ commutes with $\bar{\partial}_E$ and hence induces a chain mapping from $(C^*(u), \bar{\partial}_E)$ to $(C^*(u_0), \bar{\partial}_E)$. In particular, for $|u - u_0| \leq \epsilon$ and $u \in [a, b]$, the induced map is an isomorphism of chain complexes. In particular, the coboundaries $B^q(t)$ map isomorphically to the coboundaries $B^q(u_0)$, the cocyles to cocycles, and cohomologies to cohomologies.

For u with $|u - u_0| \leq \epsilon$, let $lh^{q,j}(u), b^{q,k}(u), \tilde{b}^{q,l}(u)$ be the unique elements of $C^{p,q}(E, K, g(u))$ mapping under the fixed spectral projection $\Pi_{E,K,g(u_0)}$ to the elements $lh^{q,j}(u_0), b^{q,k}(u_0), \tilde{b}^{q,l}(u_0)$. Then necessarily, $\bar{\partial}_E \tilde{b}^{q,l}(u) = b^{q,l}(u)$, the $\{b^{q,k}(u)\}$'s form an ordered basis for the coboundaries $B^q_p(u)$, $\bar{\partial}_E lh^{q,j}(u) = 0$ and $\{lh^{q,j}(u)\}$ project to a basis for $H^q(C^*_p(u), \bar{\partial}_E)$. Since the isomorphism $H^q(C^*_p(u), \bar{\partial}_E) = H^q(A^{p,*}(W, E), \bar{\partial}_E)$ carries $lh^{q,j}(u)$ to the cohomologous $lh^{q,j}(u_0)\}$ which represents $h^{q,j} \in H^{p,q}(W, E)$, the element $lh^{q,j}(u)$ under the isomorphism $H^q(C^*_p(u), d) = H^q(A^{p,*}(W, E), d) = H^{p,q}_{\bar{\partial}}(W, E)$ represents the same element also.

In particular, the ordered set of cocyles, $\mathbf{lh}^q(u) = \{lh_{q,j}(u) | j = 1, \cdots v_q\}$ represent cohomology classes which map to the above basis, $\mathbf{h}^q = \{h^{q,j} | j = 1 \cdots v_q\}$ of $H^{p,q}(W, E)$. Also the ordered coboundaries $\mathbf{b}^q(u) = \{b^{q,1}(u), \cdots, b^{q,s_q}(u)\}$ give an ordered basis for $B^q(u)$ and the ordered set $\tilde{\mathbf{b}}^q(u) = \{\tilde{b}^{q,1}(u), \cdots, \tilde{b}^{q,s_q}(u)\}$ $\subset C^{p,q-1}(u)$ maps under $d = \bar{\partial}_E$ to the ordered basis $\mathbf{b}^q(u)$ above.

Correspondingly for $C^{*,N-p}(K, g(u))$ and the coboundary D'', for u with $|u - u_0| \leq \epsilon$, let $lh'^{q,j}(u), b'^{q,k}(u), \tilde{b'}^{q,l}(u)$ be the unique elements of $C^{q,N-p}(E, K, g(u))$ mapping under the fixed spectral projection $\Pi_{E,K,g(u_0)}$ to the elements $lh'^{q,j}(u_0), b'^{q,k}(u_0), \tilde{b'}^{q,l}(u_0)$. Then necessarily, $D''\tilde{b'}^{q,l}(u) = b'^{q,l}(u)$, the $\{b'^{q,k}(u)\}$'s form an ordered basis for the coboundaries of D'', $D'' lh'^{q,j}(u) = 0$ and $\{lh'^{q,j}(u)\}$ project to a basis for $H^q(C^{*,N-p}(K, g(u)), D'')$. Since the isomorphism $H^q(C^{*,N-p}(K, g(u)), D'') = H^q(A^{*,N-p}(W, E), D'')$ carries $lh'^{q,j}(u)$ to the cohomologous $lh'^{q,j}(u_0)\}$ which represents $h'^{q,j} \in H^{q,N-p}_{D''}(W, E)$, the element $lh'^{q,j}(u)$ under the isomorphism $H^q(C^{*,N-p}(K, g(u)), D'') = H^q(A^{*,N-p}(W, E), D'') = H^{q,N-p}_{D''}(W, E)$ represents the same element also.

In particular, the ordered set of cocyles, $\mathbf{lh'}^q(u) = \{lh'_{q,j}(u) | j = 1, \cdots u_q\}$ represent cohomology classses which map to the above basis, $\mathbf{h'}^q = \{h'^{q,j} | j = 1 \cdots u_q\}$ of $H^{q,N-p}_{D''}(W, E)$. Also, the ordered coboundaries $\mathbf{b'}^q(u) = \{b'^{q,1}(u), \cdots, b'^{q,r_q}(u)\}$ give an ordered basis for the coboundaries and the ordered set $\tilde{\mathbf{b'}}^q(u) = \{\tilde{b'}^{q,1}(u), \cdots, \tilde{b'}^{q,r_q}(u)\} \subseteq C^{q-1,p}(u)$ maps under D'' to the

ordered basis $\mathbf{b'}^q(u)$ above.

In particular, the ordered set $(\hat{\star}_{g(u)}\otimes Id_E)\mathbf{lh'}^{N-q}(u) \subset C^{N-q,N-p}(E,K,g(u))$ is in the kernel of d^* and projects to a basis of $H^{N-q}(C^*(u),d^*)$. Similarly, the ordered set $(\hat{\star}_{g(u)}\otimes Id_E)\mathbf{b'}^{N-q}(u)$ forms an ordered basis for these boundaries. Moreover, $d^*(\hat{\star}_{g(u)}\otimes Id_E)^{-1}\tilde{\mathbf{b'}}^{N-q}(u)) = \pm(\hat{\star}_{g(u)}\otimes Id_E)\mathbf{b'}^{N-q}(u)$, so these form a suitable choice to compute the algebraic torsion for the complex $C^{p,\star}(K,g(u)),d^*)$. One sets $\{\mathbf{h'}_q(u)\} = \{[(\hat{\star}_{g(u)}\otimes Id_E)\mathbf{lh'}^{N-q}(u)]\}$ in $H^q(C^*(u),d^*)$. Under the isomorphism

$$H^q(C^*(u),d^*) \cong H_{D''}^{N-q,N-p}(W,E)$$

described above, this basis corresponds to the chosen basis $\mathbf{h'}^{N-q}$ of $H_{D''}^{N-p,N-q}(W,E)$, which is independent of u.

With these choices one is able to write the algebraic torsion as a function of u for $|u-u_0|\le \epsilon$. With the short hand $\hat{\star}_{g(u),E} = (\hat{\star}_{g(u)}\otimes Id_E)$ it is:

$$\begin{aligned}
&torsion'(C_p^*(t),d,d^*,\{\mathbf{h}^q(u)\},\{\mathbf{h'}_q(u)\})\\
&= (-1)^S\ \prod_{q=0}^N [\{\mathbf{b}^q(u)\},\{\mathbf{lh}^q(u)\},\{\tilde{\mathbf{b}}^{q+1}(u)\}/\\
&\hat{\star}_{g(u),E}\{\mathbf{b'}_{N-q}(u)\},\hat{\star}_{g(u),E}\{\mathbf{lh'}^{N-q}(u)\},\hat{\star}_{g(u),E}\{\tilde{\mathbf{b'}}^{N-(q+1)}(u)\}]^{(-1)^{q+1}}
\end{aligned}$$

for a fixed sign $(-1)^S$.

This may be computed via the above device.

For example, if the ordered basis $\{\mathbf{b}^q(u)\},\{\mathbf{lh}^q(u)\},\cdots,\{\tilde{\mathbf{b}}^{q+1}(u)\}$ is $[e_1,\cdots,e_L]$ and the ordered basis $\hat{\star}_{g(u),E}\{\mathbf{b'}_{N-q}(u)\},\hat{\star}_{g(u),E}\{\mathbf{lh'}^{N-q}(u)\},\hat{\star}_{g(u),E}\{\tilde{\mathbf{b'}}^{N-(q+1)}(u)\}$ is $[f_1,\cdots,f_L]$, then the required determinant is a where $(\wedge e_j) = a(\wedge\ f_k)$. The required number a can be computed by just taking the projection onto the fixed summand $\wedge^L\ C^{p,q}(E,K,g(u_0))$ by means of the spectral projection $\Pi_{E,K,g(u_0)}$.

By the above choices one has

$$\begin{aligned}
&b^{q,j}(u) = b^{q,j}(u_0) + v\ for\ some\ v\in P_q(u_0)\\
&\tilde{b}^{q,k}(u) = \tilde{b}^{q,j}(u_0) + v\ for\ some\ v\in P_q(u_0)\\
&lk^{q,l}(u) = lk^{q,l}(u_0) + v for\ some\ v\in P_q(u_0)
\end{aligned}$$

Thus, the projection of the wedge product of the terms in the ordered basis $\{\mathbf{b}^q(u)\},\{\mathbf{lh}^q(u)\},\{\tilde{\mathbf{b}}^{q+1}(u)\}$ is precisely the projection wedge product of the entries in the reference ordered basis $\{\mathbf{b}^q(u_0)\},\{\mathbf{lh}^q(u_0)\},,\{\tilde{\mathbf{b}}^{q+1}(u_0)\}$. In particular, the derivative of this projection by u is zero.

By the above choices,

$$\begin{aligned}
&b'^{q,j}(u) = b'^{q,j}(u_0) + v\ for\ some\ v\in P_{N-p}(u_0)\\
&\tilde{b'}^{q,k}(u) = \tilde{b'}^{q,j}(u_0) + v\ for\ some\ v\in P_{N-p}q(u_0)\\
&lk'^{q,l}(u) = lk'^{q,l}(u_0) + v for\ some\ v\in P_{N-p}(u_0)
\end{aligned}$$

Hence, since $\hat{\star}_{g(u_0),E}$ commutes with $\Delta_{E,\bar{\partial},g(u_0)}$ and also the fixed projection

$\Pi_{E,K,g(u_0)}$,

$$\begin{aligned}
&\hat{\star}_{g(u_0),E}{b'}^{q,j}(u) = \hat{\star}_{g(u_0),E}{b'}^{q,j}(u_0) + w \ for\ some\ w \in P_q(u_0)\\
&\hat{\star}_{g(u_0),E}\tilde{b'}^{q,k}(u) = \hat{\star}_{g(u_0),E}\tilde{b'}^{q,j}(u_0) + w \ for\ some\ w \in P_q(u_0)\\
&\hat{\star}_{g(u_0),E}lk'^{q,l}(u) = \hat{\star}_{g(u_0),E}lk'^{q,l}(u_0) + w for\ some\ w \in P_q(u_0)
\end{aligned}$$

So in particular,

$$\begin{aligned}
&d/du\ \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}{b'}^{q,j}(u)|_{u=u_0} = d/du\ \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}{b'}^{q,j}(u_0)|_{u=u_0}\\
&d/du\ \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\tilde{b'}^{q,k}(u)|_{u=u_0} = d/du\ \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\tilde{b'}^{q,k}(u_0)|_{u=u_0}\\
&d/du\ \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}lk'^{q,l}(u)|_{u=u_0} = d/du\ \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}lk'^{q,l}(u_0)|_{u=u_0}
\end{aligned}$$

Consequently, by these observations, one has the simplification for $|u-u_0| \leq \epsilon$,

$$\begin{aligned}
&d/du\ torsion'(C_p^*(u), d, d^*, \{\mathbf{h}^q(u)\}, \{\mathbf{h'}_q(u)\})|_{u=u_0}\\
&= (-1)^S\ d/du\ \textstyle\prod_{q=0}^N\ [\{\mathbf{b}^q(u_0)\}, \{\mathbf{lh}^q(u_0)\}, \{\tilde{\mathbf{b}}^{q+1}(u_0)\}/\\
&\quad \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\{\mathbf{b'}_{N-q}(u_0)\},\\
&\qquad \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\{\mathbf{lh'}^{N-q}(u_0)\}\\
&\qquad\quad , \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\{\tilde{\mathbf{b'}}^{N-(q+1)}(u_0)\}]^{(-1)^{q+1}}\ |_{u=u_0}
\end{aligned}$$

since the above method of computation yields the same result for both expressions.

This last gives

$$\begin{aligned}
&d/du\ \left[\frac{torsion'(C^{p,*}(u), d, d^*, \{\mathbf{h}^q(u)\}, \{\mathbf{h'}_q(u)\})}{torsion'(C^{p,*}(u_0), d, d^*, \{\mathbf{h}^q(u_0)\}, \{\mathbf{h'}_q(u_0)\})}\right]^{-1}\ |_{u=u_0}\\
&= d/du\ \textstyle\prod_{q=0}^N\ [\Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\{\mathbf{b'}_{N-q}(u_0)\},\\
&\qquad \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\{\mathbf{lh'}^{N-q}(u_0)\}, \Pi_{E,K,g(u_0)}\hat{\star}_{g(u),E}\{\tilde{\mathbf{b'}}^{N-(q+1)}(u_0)\}/\\
&\quad \hat{\star}_{g(u_0),E}\{\mathbf{b'}_{N-q}(u_0)\}, \hat{\star}_{g(u_0),E}\{\mathbf{lh'}^{N-q}(u_0)\}\\
&\qquad , \hat{\star}_{g(u_0),E}\{\tilde{\mathbf{b'}}^{N-(q+1)}(u_0)\}]^{(-1)^{q+1}}\ |_{u=u_0}
\end{aligned}$$

Consequently, one gets

Lemma 6.1 *Under the above hypotheses:*

$$\begin{aligned}
&d/du\ \log\ torsion(C_p^*(u), d, d^*, \{\mathbf{h}^q(u)\}, \{\mathbf{h'}_q(u)\})|_{u=u_0}\\
&= -\ \Sigma_{q=0}^n(-1)^q Tr(\Pi_{E,K,g(u_0)}(\alpha_{g(u_0)} \times Id_E)|V(p, q, E, K, g(u_0)))
\end{aligned}$$

with $\alpha_{g(u)}$ *the bundle mapping* $\alpha_{g(u)} = (\hat{\star}_{g(u)})^{-1}\ d/du\ (\hat{\star}_{g(u)}) =$
$-\ d/du\ (\hat{\star}_{g(u)})\ (\hat{\star}_{g(u)})^{-1}$

7 Definition of the flat complex analytic torsion $\tau(M, F)$

Let M be a closed, Riemannian, smooth manifold of dimension n. Let the metric on TM, the tangent bundle of M be denoted by $g : TM \times TM \to R$. For convenience we assume M is oriented although this is not necessary.

Suppose that $F \to M$ is a flat k dimensional complex vector bundle. That is, the transition matrices of F are locally constant (but not necessarily unitary).

The considerations above carry over almost verbatim to this smooth setting when the d-bar Laplacian above $\Delta^\flat_{E,\bar\partial}$ is replaced by the flat Laplacian $\Delta^\flat_F$. The result is the metric independence of the analytic torsion $\tau(M,E)$ of the metric for flat bundle E over an odd dimensional Riemannian manifold M. In this section we merely state the parallel results. The proofs are quite the same in detail.

Utilizing the method of section 2, the exterior derivative d and its adjoint d^* and the standard Laplacian $\Delta = (d+d^*)^2 = dd^* + d^*d$ acting on complex valued forms $A^*(M,C)$ have canonical extensions, called here $d^\flat_F$, $d^{*,\flat}_F$ and $\Delta^\flat_E$ with

$$\Delta^\flat_E = (d^\flat_F + d^{*,\flat}_F)^2 = d^\flat_F d^{*,\flat}_F + d^\flat_F d^{*,\flat}_F$$

and commuting with $(d^\flat_F)^2 = 0, (d^{*,\flat}_F)^2 = 0$.

Traditionally, $d^\flat_F$ is denoted by d_F, so this convention is followed here.

More explicitly, let $A^p(M,C)$ be the smooth forms on M with complex values, i.e., smooth sections of the exterior power $\wedge^i(T^*M)$ of the cotangent bundle T^*M, $A^p(M,C) = \Gamma(\wedge^i(T^*M)_R \otimes C)$. The exterior derivative d defines a natural first–order operator

$$d : A^p(M,C) \to A^{p+1}(M,C) \qquad (7.48\ )$$

with $d^2 = 0$, giving the deRham complex $(A^*(M,C), d)$.

Let $A^p(M,F)$ denote p-forms with values in the flat bundle F, i.e.,$A^p(M,F) = \Gamma(\wedge^i(T^*M)_R \otimes F)$. Since F is a flat bundle, with constant transition matrices, there is a unique first–order differential operator

$$d_F\{= d^\flat_F\} : A^p(M,F) \to A^{p+1}(M,F)$$

with the properties $d_F a \otimes s = da \otimes s$ over a open set U for any complex valued p-form a and flat section s of F over U. One easily sees that $d_F^2 = 0$. The deRham theorem in this context asserts that integration of forms gives a natural isomorphism of cohomologies:

$$H^p(A^*(M,F), d_F) \xrightarrow{\cong} H^p(M,F)$$

where the last is the topologically defined cohomology with coefficients in the flat coefficient system F.

The chosen Riemannian metric g determines a unique star operator $\star$ from complex valued p-forms to $n-p$ forms. It is induced by a bundle isomorphism:

$$\Lambda^p T^*M \xrightarrow{\star} \Lambda^{N-p} T^*M \qquad (7.49\ )$$

with $< a(x), b(x) > \ dvol_x = a(x) \wedge \star b(x)$ at any point $x \in W$. In terms of local coordinates $\{x_1, \cdots x_n\}$ in a neighborhood of the point $x \in W$ for which $\{dx_j\}$ at x is an orthonormal basis of T^*M under g, one has at x

$$\star\ [(dx_{i_1} \wedge \cdots \wedge dx_{i_p})]|_x = \ sign\ [(dx_{k_1} \wedge \cdots \wedge dx_{k_{n-p}})]|_x$$

where $\{1,2,\cdots,N\} = \{i_1,\cdots,i_p\} \sqcup \{k_1,\cdots,k_{N-p}\}$. Here the sign is the sign of the permutation $[1,2,\cdots,n] \mapsto [i_1,\cdots,i_p,k_1,\cdots,k_{n-p},l'_1]$. Note $\star^2|_{A^p(M,C))} = (-1)^{p(n-p)}\ Id$.

In particular, a linear star isomorphism of forms with values in F is induced by $\star \otimes Id_F$:

$$A^p(M,F) = \Gamma(\Lambda^p T^M \otimes F) \overset{(\star\ \otimes\ Id_F)}{\rightarrow} \Gamma(\Lambda^{n-p}T^*M \otimes F) \\ = A^{n-p}(M,F)$$

In these terms the first–order differential operator $d_F^{*,\flat} : A^p(M,F) \rightarrow A^{p-1}(M,F)$ is the composite:

$$(-1)^{np+n+1}\ (\star \otimes Id_F)\ d_F\ (\star \otimes Id_F) \quad : \tag{7.50 }$$

Here by definition $(d_F^{*,\flat})^2 = 0$.

Consequently, one has a complex. $A^*(M,F)$ with two differential $d_F, d_F^{*,\flat}$ of degrees $+1$ and -1 so the considerations of sections 3 to 5 apply with but small changes.

It is important to note that if a smooth Hermitian inner product, say $< .,. >_F$ is chosen on F, then the adjoint of d_F is **not** in general the above mapping. Rather, as specified by Müller [26], the adjoint is defined in terms of the induced conjugate linear bundle isomorphism $\mu : F \rightarrow F^*$, of F and its dual F^* as:

$$\delta_F = (-1)^{np+n+1}(\star \otimes Id_F)\ \mu^{-1}\ d_F\ \mu\ (\star \otimes Id_F) \tag{7.51 }$$

With this definition $< d_F\ s, t >_F = < s, \delta_F\ t >_F$ for the induced inner product on forms with values in F. In most of the literature, this adjoint is rather sloppily written as $d^* = (-1)^{np+p+1} \star d_F \star$ for suitable $\star$.

In particular, defining the flat Laplacian $\Delta_F^\flat$ by

$$\Delta_E^\flat = (d_F + d_F^{*,\flat})^2 = d_F\ d_F^{*,\flat} + d_F^{*,\flat}\ d_F \tag{7.52 }$$

yields a generally non-self adjoint operator, as opposed to the standard self-adjoint Riemannian Laplacian coupled to E, Δ_E, defined by

$$\Delta_F = (d_F + d_F^{*,\flat})^2 = d_F\ d_F^{*,\flat} + d_F^{*,\flat}\ d_F. \tag{7.53 }$$

It is apparent that $\Delta_F^\flat$ has a simpler definition than the classical self-adjoint Δ_F which uses the additional information of a choice of Hermitian inner product, $< .,. >_F$, on F. In contrast the operator $\Delta_E^\flat$ utilizes only the Riemannian inner product g on TM.

However, if F is a unitarily flat bundle then obviously:

$$\Delta_F^\flat = \Delta_F \textit{ when } F \textit{ is unitarily flat} \tag{7.54 }$$

On the other hand, the symbols of the two first–order operators $\delta_F^\flat$ and $d_F^{\star,\flat}$ are identical, so they differ by a smooth bundle mapping, say B,

$$d_F^{\star,\flat} = \delta_F + B.$$

In particular, $\Delta_F^\flat = \Delta_F + (Bd_F + d_F B)$,so the generally non-self adjoint operator $\Delta_F^\flat$ has the same symbol as the second–order self adjoint operator Δ_F, both acting on $A^p(M,F)$.

Consequently, standard methods of elliptic theory apply, e.g., the methods of Atiyah, Patodi, Singer and Seeley, [2, 33, 34].

As before, one has the theorem that the generalized eigenvalues of $\Delta_F^\flat$ lie in a fixed parabola about the positive real axis.

Let $K > 0$ be a real number which is not the real part of any eigenvalue. Let $S(p,K)$ be a complete enumeration of all the generalized eigenvalues counted with multiplicities with real part greater than K. That is, $\Re(\lambda_j) > K$.

Then again the zeta function

$$\zeta_{F,K,g}(s) = \Sigma_{\lambda_j \in S(p,K)} \ \frac{1}{\lambda_j^s} \tag{7.55 }$$

defined using the principle part of the powers converges for $Re(s) > N$. If $\Pi_{F,K,g}$ denotes the spectral projection on the span of the generalized eigenvectors with generalized eigenvalues with real part less than K, then the elliptic operator $(1 - \Pi_{F,K,g})\ \Delta_F^\flat$ fits the setting of Seeley [33, 34]. In particular, its complex powers $[(1 - \Pi_{F,K,g})\ \Delta_F^\flat]^{-s}$ are well defined as in that paper. Also as in [34], the "heat kernel" $e^{-t\ (1-\Pi_{F,K,g})\ \Delta_F^\flat}$ is well defined for $t > 0$ real, it is of trace class, and for $Re(s) > N$ the formula

$$[(1 - \Pi_{F,K,g})\ \Delta_F^\flat]^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \ t^{s-1}\ e^{-t\ (1-\Pi_{F,K,g})\ \Delta_F^\flat}\ dt \tag{7.56 }$$

holds and its trace gives the zeta function $\zeta_{F,K,g}(s)$ for the bundle F.

Moreover, these methods of Seeley imply that $\zeta_{F,K,g}(s)$ has a meromorphic extension to the whole plane and is analytic at $s = 0$. Consequently, the derivative at $s = 0$, $\zeta'_{F,K,g}(0)$ is meaningful.

Following Ray and Singer [31] and the above, one sets

$$Ray - Singer - Term(F,K,g) \tag{7.57 }$$

$$= exp((1/2)\ \Sigma_{q=0}^N\ (-1)^q\ q\ \zeta'(F,K,g)(0)) \tag{7.58 }$$

In view of the fact that d_F and $d_F^{*,\flat}$ commute with $\Delta_F^\flat$ the generalized eigensolutions with eigenvalue $\lambda \neq 0$ in the image of $d_F^{*,\flat}$ map by d_F isomorphically onto those in the image of d_F. As in Ray and Singer [30], this implies

complete cancelation in equation 7.57 if the dimension of M is even. Hence, $Ray - Singer - Term(F, K, g) = 0$ in this case.

Consequently, in the remaining part of this section it is assumed that M is a closed **odd** dimensional smooth manifold. For n odd one has the equations:

$$(\star \otimes Id_F)^2 = Id \text{ and}$$
$$d_F^{*,\flat}|_{A^p(M,F)} = (-1)^p \ (\star \otimes Id_F) d_F (\star \otimes Id_F)$$

In this setting the methods of Ray and Singer [30] again apply to prove the following lemma.

Lemma 7.1 *Suppose the smooth closed manifold M has odd dimension n. Fix $K > 0$. Let g_t, $a \leq t \leq b$ be a smooth family of Riemannian metrics on TM such that the operators $\Delta^\flat_F, g_t$ have no generalized eigenvalues with real parts equal to K. Here the dependence on g_t is explicitly recorded.*

Let the star operator $\star_t$: be the bundle isomorphism associated to the Riemannian inner product g_t.

$$\Lambda^p T^* M \xrightarrow{\star_t} \Lambda^{n-p} T^* M \tag{7.59 }$$

Let $\Pi_{F,K,t}$ be the spectral projection onto the span of the generalized eigenvectors of $\Delta^\flat_{F,g_t}$ with generalized eigenvalues of real part less than K.

Then $Ray{-}Singer{-}Term(F, K, g) = exp((1/2)\ \Sigma_{q=0}^N \ (-1)^q \ q \ \zeta'(F, K, g_t)(0))$ varies smoothly with t for $a \leq t \leq b$ and similarly for C^k and moreover,

$$\begin{aligned} &d/dt \ log \ det \ Ray - Singer - Term(F, K, g_t) \\ &\quad = (1/2) \ Tr(\Pi_{F,K,g_t} \ (\alpha_t \otimes Id_F)) \end{aligned} \tag{7.60 }$$

where $\alpha_t = \star_t^{-1}(d/dt \ \star_t)$, a bundle mapping acting on $\Lambda^*(T^*M)$.

Similarly, the methods of Ray and Singer straight-forwardly give the expected dependence of the $Ray - Singer - Term(F, K, g)$ on changing $K > 0$. It is:

Lemma 7.2 *Fix $L > K > 0$. Let g be a Riemannian metric on TM such that the operator $\Delta^\flat_{F,g}$ acting on $A^*(M, F)$ has no eigenvalues with real part equal to K or L. Let $\{\lambda_{p,j} | j = 1 \cdots n_p\}$ be a complete enumeration counting with multiplicities of the generalized eigenvalues of $\Delta^\flat_{F,g}$ acting on $A^p(M, F)$ which have real part in the range K to L. Then the formula below holds:*

$$\left[\frac{Ray - Singer - Term(F, K, g)}{Ray - Singer - Term(F, L, g)}\right]^2 = \left[\prod_{p=0}^{N} (\prod_{j=1}^{n_p} \ \lambda_{p,j})^{(-1)^p \ p}\right]^{-1} \tag{7.61 }$$

Let $V(F, K, g)$ denote the span in $A^p(M, F)$ of the generalized eigensolutions of $\Delta^\flat_{F,g}$ of with generalized eigenvalues with real part less than K. By elliptic theory $V(F, K, g)$ is a finite dimensional subspace of smooth sections.

In view of the commutation relations,

$$\begin{array}{lll} \Delta^\flat_{F,g} d_F & = d_F \ d_F d_F^{*,\flat} \ d_F & = d_F \ \Delta^\flat_{F,g} \\ and & (\star \otimes Id_F) \ \Delta^\flat_{F,g} & = \Delta^\flat_{F,g} \ (\star \otimes Id_F) \end{array}$$

the graded complex

$$C^*(F,K,g) := \bigoplus_{p=0}^{N} V(p,F,K,g)$$

has two differentials, d_F and $d_F^{*,\flat} = (-1)^{np+n+1} \ (\star \otimes Id_F) d_F (\star \otimes Id_F)$. Here $(-1)^{np+n+1} = (-1)^p$, $d_F : V(p,F,K,g) \to V(p+1,F,K,g)$ increases grading by one and $d_F^{*,\flat} : V(p,F,K,g) \to V(p-1,F,K,g)$ decreases degree by one. That is, one has the same complex equipped with two differentials, a bi-complex:

$$(C^*(F,K,g), d_F, d_F^{*,\flat}) \tag{7.62 }$$

of degrees $+1, -1$ respectively.

In §5, it was proved that in this algebraic setting of a graded complex of length n, C^* with two differentials, d of degree $+1$ and d^* of degree -1, i.e, (C^*, d, d^*), there is a natural non-vanishing algebraic torsion invariant

$$torsion(C^*, d, d^*) \in (det \ H^*(C^*, d)) \otimes (det \ H_*(C^*, d^*))^* \ .$$

The present example, $(C^*(F,K,g), d_F, \delta^\flat_F)$ fits this pattern. Hence, one gets a non-vanishing torsion invariant

$$\begin{array}{l} torsion((C^*(F,K,g), d_F, \delta^\flat_F) \\ \qquad \in (det \ H^*(C^*_p(F,K,g), d_F) \otimes (det \ H_*(C^*_p(F,K,g), d_F^{*,\flat}))^* \end{array}$$

These cohomologies and homologies can be identified as follows:

Let N be the multiplicity of the generalized eigenvalue 0 for $\Delta^\flat_F$ and $\mathcal{H}$ denote the span of these generalized eigenvectors with generalized eigenvalue 0. By spectral theory $\mathcal{H}$ is finite dimensional and consists of smooth sections. Then one simply has the direct sum decompositions:

$$A^*(M,F) = \mathcal{H} \oplus d_F[(\Delta^\flat_F)^M A^*(M,F)] \oplus \delta^\flat_F[(\Delta^\flat_F)^N \ A^*(M,F)]$$

and d_F maps the second summand isomorphically to the first. By this means one can easily show that the inclusion of co-chain complexes

$$(C^*(F,K,g), d_F) \subset (A^*(M,F), d_F)$$

induces an isomorphism on cohomology. That is, $H^q(C^*(F,K,g), d_F) \cong H^p(M,F)$. In particular, $det \ H^*(C^*(F,K,g), d_F) \cong det \ H^*(M,F)$ where $H^*(M,F)$ is the topological cohomology of the local coefficient system F.

Similarly, by $(\star \otimes Id_F)^2|_{A^p(M,F)} = \pm Id$, it follows that $(\star \otimes Id_F)(\delta^\flat_F)(\star \otimes Id_F)^{-1} = \pm d_F$; so $(\star \otimes Id_F)$ induces a complex linear isomorphism of the graded complex $C^*(F,K,g)$ to the complex $C^*(F,K,g)$ sending $C^p(M,F) \to C^{n-p}(M,F)$ with the differential $\delta^\flat_F$ becoming $\pm d_F$.

In these terms, the star operator $(\star \otimes Id_F)$ establishes a complex linear isomorphism of homologies with cohomologies:

$$\begin{aligned} & H_r(C^*(F,K,g), d_F^{*,\flat}\ ) \\ & \cong H^{n-r}(C^*(F,K,g), d_F) \\ & \cong H^{n-r}(M,F) \end{aligned}$$

In particular, one has a complex linear isomorphism of determinants:

$$det\ H_*(C^*(F,K,g), \delta^\flat_F) = (det H^*(M,F))^{(-1)^n}$$

Hence, in this situation the algebraic invariants may be regarded as an element of the same complex line,

$$\begin{aligned} & torsion(C^*(F,K,g), d_F, \delta^\flat_F), \\ & \qquad \in\ (det\ H^*(M,F)) \otimes [det\ H^*(M,F)]^{(-1)^{n+1}} \end{aligned} \tag{7.63 }$$

for any real number $K > 0$ and any choice of Riemannian metric g on TM. Additionally, using Poincaré duality one has $det\ H^{n-*}(M,F)^* \cong det\ H^*(M,F^*)$.

For the case of interest n is odd, so the algebraic torsion takes its values in the square of a determinant line bundle:

$$\begin{aligned} & torsion(C^*(F,K,g), d_F, \delta^\flat_F), \\ & \in\ (det\ H^*(M,F))^2 \cong det H^*(M, F \oplus F^*) \end{aligned} \tag{7.64 }$$

Now let $L > K > 0$ be real and positive. It follows from the algebraic lemmas B,C of §5, that under the assumptions of lemma 7.2, the algebraic torsion of $(C^*(F,K,g), d_F, \delta^\flat_F)$ and that for L with $K > L > 0$ are precisely related by the eigenvalues $\lambda_{q,j}$ of lemma 7.2 as follows:

$$\begin{aligned} & torsion(C^*(F,L,g), d_F, \delta^\flat_F)) \\ & = torsion(C^*(F,K,g), d_F, d_F^{*,\flat})) \\ & \cdot \left[\prod_{p=0}^{N} (\prod_{j=1}^{n_p}\ \lambda_{p,j})^{(-1)^p\, p} \right]^{-1} \end{aligned} \tag{7.65 }$$

when regarded as an element of $(det\ H^{p,*}(M,F))^2$. Here n is odd.

Similarly, it follows just as in the proof of lemma 6.1 of §6 for a smooth family of Riemannian metrics g_t satisfying the assumptions of lemma 7.1 one gets the equality

$$\begin{aligned} & d/dt \left[\ log\ det\ torsion(C^*(F,L,g_t), d_F, d_F^{*,\flat})\right] \\ & \qquad = -\ Tr(\Pi_{F,K,g_t}\ (\alpha_{g_t} \otimes Id_F)) \end{aligned} \tag{7.66 }$$

As an immediate consequence of the above two equations , the main theorem below follows:

Theorem 7.3 (Independence of metric theorem) *Let M be closed, smooth, oriented and of odd dimension n. For any choice of Riemannian inner product g on TM, pick a real number $K > 0$ for which the operators $\Delta^{\flat}_{F,g}$ acting on $A^*(M,F)$ have no eigenvalue with real part equal to K and similarly for the trivial bundle C^k. Define the graded complex $C^*(F,K,g) = \bigoplus_{p=0}^{N} V(p,F,K,g)$ with its two differentials, d, d^* as above and similarly for C^k. Then the combination*

$$\begin{aligned}\tau_{analytic}(M,F) \;:=\; & torsion((C^*(F,K,g), d_F, d_F^{*,\flat}) \\ & \cdot Ray - Singer - Term(F,K,g)^2\end{aligned}$$

is independent of the choice of $K > 0$ and also independent of the choice of Riemannian inner product g on TM chosen.

By definition $\tau(W,F)$ is a non-vanishing element of the squared determinant line bundle

$$\tau(M,F) \;\in (det\ H^*(M,F))^2 \cong detH^*(M, F \oplus F^*)$$

Addendum 1): *In the acyclic cases the torsion is a complex number.*

Addendum 2): *In the case that F is acyclic and flat unitary, this torsion is a real number.*

Since the operator $\Delta^{\flat}_F$ is self-adjoint in this case, one may take $K > 0$ less than the smallest non-vanishing real eigenvalue. For this choice of $K > 0$, the algebraic correction term is just $+1$. For F flat unitary and acyclic $\tau(W,F)$ is identical with the square of Ray Singer analytic torsion [30] and by the Cheeger-Müller theorem [19, 25] equals the square of the Riedemeister-Franz topological torsion in this special flat unitary acyclic case.

As explained, the proof is completely parallel to the d-bar case, except there is no global anomaly term since the dimension n is odd.

8 Combinatorial torsion for general flat bundles over compact smooth manifolds

Let F be a flat bundle over a general compact smooth oriented manifold M, possibly with boundary. In this section a "Reidemeister- Franz" type of torsion

$$\tau_{comb}(M,F) \in det\ H^*(M, F \oplus F^*)$$

is defined in a combinatorial fashion, an invariant of the pair (M,F). Here $F^* = Hom(F,C)$ is the dual bundle to F. The method is very straightforward.

Let F be a flat complex bundle over an oriented n–dimensional manifold with a cellular subdivision, say $W = \{\sigma\}$ with dual cellular complex, say $D(W) = \{D(\sigma\})$. Here for σ an oriented cell, let $D(\sigma)$ is the dual cell with

the corresponding compatible orientation. Let $K(\sigma)$ be the flat sections over σ and $K(D(\sigma))$ be the flat sections over $D(\sigma)$. One gets a **canonical** isomorphism, by restricting to the common center:

$$K(\sigma) \cong K(D(\sigma))$$

Hence, summing over oriented cells and dually oriented cells, we get a canonical isomorphism summing over cells of dimension a

$$\Theta : C^a(W, F) = \oplus\, K(\sigma) \cong \oplus\, K(D(\sigma)) = C^{n-a}(D(W), F)$$

Let d, d' be the coboundaries for W, $D(W)$ respectively.

By means of Θ one may identify these, and so endow $C^*(W, F)$ with two differentials on the same complex $C^*(W, F)$:

$$d, \; going\ up, \quad and\ \delta = \Theta^{-1}\, d'\, \Theta, \quad going\ down$$

and so define an algebraic torsion $\tau_{comb.}(C^*(W, F), d, \delta)$ as in section 5. By methods similar to those used in Milnor [24] $\tau_{comb.}(C^*(W, F), d, \delta)$, can be shown to be independent of the choice of cellular decomposition.

$$\tau(M, F) = \tau_{comb.}(C^*(W, F), d, \delta)$$

The cohomology of the cell complex W gives $H^*(M, F)$; the cohomology of the cell complex $D(W)$ gives the relative cohomology $H^*(M, \partial M, F)$.

This element, $\tau(M, F)$, by Poincaré duality lies in

$$\begin{aligned} &\tau(M, F) \in \; det(H^*(W, F)) \otimes det(H^{n-*})(D(W), F)^* \\ &\cong det(H^*(M, F)) \otimes det(H^{n-*}(M, \partial M, F))^* \\ &\cong det(H^*(M, F)) \otimes det(H^*(M, F^*))^* \\ &\cong det(H^*(M, F \oplus F^*)) \end{aligned}$$

Note on the non-oreinted case: In the case M is not oriented, then the isomorphism Θ becomes an isomorphism, $\Theta' : C^a(W, F) \cong C^{n-a}(D((W), F \otimes O)$ where O is the orientation sheaf. In this case the resultant torsion lies in $det(H^*(W, F)) \otimes det(H^{n-*})(D(W), F \otimes O)^* \cong det(H^*(M, F)) \otimes det(H^{n-*}(M, \partial M, F \otimes O))^* \cong det(H^*(M, F)) \otimes$
$det(H^*(M, F^*)) \cong det(H^*(M, F \oplus F^*))$ again.

A more functorial description of this element $\tau(M, F) = \tau_{comb.}(C^*(W, F), d, \delta)$ is as follows: We assume for convenience M is oriented.

The isomorphism Θ^{-1} defines an element

$$[\Theta^{-1}] \in (det\ C^*(W, F)) \otimes (det\ C^{n-*}(D(W), F))^*$$

so under the Bismut-Zhang isomorphisms [8], $\tau \otimes \tau$, one obtains an element

$$\begin{aligned} &(\tau \otimes \tau)[\Theta^{-1}] \in (det\ H^*(W, F)) \otimes (det\ H^{n-*}(D(W), F))^* \\ &= (det\ H^*(M, F)) \otimes (det\ H^{n-*}(M, \partial M, F))^* \cong det\ H^*(M, F \oplus F^*) \end{aligned}$$

where the last step used Poincaré duality. In these terms one has

$$\tau_{comb}(M,F) = \tau_{comb.}(C^*(W,F),d,\delta) = (-1)^{S(C)}\ (\tau\otimes\tau)[\Theta^{-1}]$$

with the sign $(-1)^{S(C)}$ as above.

As a special case of this construction, we may take a Morse function with its associated downwards and upwards cells.

9 Comparison of Combinatorial and Analytic Torsions: Generalization of the Cheeger–Müller theorem:

Let F be a k dimensional flat complex bundle over a smooth oriented odd dimensional Riemannian manifold M of dimension n with Riemannian metric g. Let

Chose $K > 0$ as in section 7 so it is not the real part of any eigenvalue of the flat Laplacian $\Delta_F^\flat$ and let $V^*(F,K,g)$ the span of the smooth forms which have eigenvalues with real part less than or equal to K.

Now d_F and $\hat{\star} d_F \hat{\star}$ both commute with $\Delta_F^\flat$, and $V^*(F,K,g)$ acquires by restriction two differentials and so one may form the associated algebraic torsion as in §5. The torsion $\tau_{analytic}(M,F,g)$ is obtained from this by multiplying by the square of the Ray-Singer term.

A more functorial way to express this analytic torsion is as follows: Consider the star operator $\hat{\star} = \star\otimes Id_F$, which gives an isomorphism:

$$\hat{\star} : V^*(F,K,g) \cong V^{h-*}(F,K,g)$$

This isomorphism defines an element

$$[\hat{\star}] \in (det\ V^*(F,K,g)) \otimes (det\ V^{n-*}(F,K,g))^*$$

Now using the differentials, d_F, d_F on each, by the Bismut-Zhang torsion isomorphism τ [8], one obtains an element:

$$(\tau\otimes\tau)[\hat{\star}] \in (det\ H^*(V^*(F,K,g),d_F)) \otimes (det\ H^{n-*}(V^*(F,K,g),d_F)^*$$

Now integrating these forms over the manifold, by the deRham theorem gives an isomorphism:

$$\begin{aligned}&\textstyle\int\otimes\int : (det\ H^*(V^*(F,K,g),d_F)) \otimes (det\ H^{n-*}(V^*(F,K,g),d_F)^* \\ &\cong (det\ H^*(M,F) \otimes (det\ H^{n-*}(M,F)^* \cong det\ H^*(M,F\oplus F^*)\end{aligned}$$

From this perspective, the analytic torsion $\tau_{analytic}(M,F,g)$ is defined by

$$\begin{aligned}\tau(M,F,g) \quad &= (-1)^{S(C)}\ [Ray-Singer-Term(p,E,K,g)]^2 \\ &\times(\textstyle\int\otimes\int)[(\tau\otimes\tau)[\hat{\star}]] \in det\ H^*(M,F\oplus F^*)\end{aligned}$$

As observed in section 7, it is independent of the metric g.

Theorem 9.1 (Generalized Cheeger Müller Theorem) *For a closed, oriented, odd dimensional manifold M, the two torsion elements $\tau_{analytic}(M, F, g)$ and $\tau_{comb}(M, F)$ are equal in det $H^*(M, F \oplus F^*)$.*

This theorem was a conjecture in an earlier version of this manuscript. Following the methods of Su and Zhang, a direct proof is given below. Note that for the special case where F admits a smoothly varying non-degenerate inner product, a mild constraint, this theorem was proved by Braverman and Kappeller [11]. However, the following approach has a certain attractive naturality.

Chose a Morse function f on M which satisfies the Morse-Smale conditions with critical points the finite set B, downward or descending cell decomposition $W^u = \{W_i^u\}$ and dual upward or ascending cell decomposition $D(W_u) = W^s = \{W_i^s\}$, $W_i^s = D(W_i^u)$. Then the integration maps and evaluation

$$\begin{aligned}&\int \otimes \int : (det\ H^*(V^*(F, K, g), d_F)) \otimes (det\ H^{n-*}(V^*(F, K, g), d_F)^* \\ &\cong (det\ H^*(M, F) \otimes (det\ H^{n-*}(M, F)^* \cong det\ H^*(M, F \oplus F^*)\end{aligned}$$

can be concretely realized by integrating over the cells and dual cells as in:

$$\begin{aligned}&\int \otimes \int : (det\ H^*(V^*(F, K, g), d_F)) \otimes (det\ H^{n-*}(V^*(F, K, g), d_F)^* \\ &\cong (det\ H^*(C^*(W^u, F) \otimes (det\ H^{n-*}(C^*(D(W^u), F)^* \cong det\ H^*(M, F \oplus F^*)\end{aligned}$$

This allows the reformulation of theorem 9.1 as a comparison result about the relation of the star operator on forms and the duality mapping Θ of cells to dual cells.

Theorem 9.2 (Analytic - Geometric version:) *The element $(-1)^{S(C)}$ $[Ray- Singer - Term(p, E, K, g)]^2$ $[\hat{\star}^{-1}]$ regarded as an element of*
$(det\ V^*(K, F, g)) \otimes (det\ V^{N-*}(V, F, K))^* \overset{\tau \otimes \tau}{\cong}$
$(det\ H^*(V^*(K, F, g), d_F)) \otimes (det\ H^{n-*}(V^*(V, F, K), d_F))^*$ *passes by integration,* $\int \otimes \int$ *to the element of*
$(det\ H^*(C^*(W^u, F)) \otimes (det\ H^{n-*}(C^*(D(W^u), F))^*$ *given by the duality mapping* $(-1)^{S(C')}$ $[\Theta^{-1}]$ *regarded as an element of*
$(det\ C^*(W^u, F)) \otimes (det\ (C^{n-*}(D(W^u), F))^* \overset{\tau \otimes \tau}{\cong}$
$(det\ H^*(C^*(W^u, F)) \otimes (det\ H^{n-*}(C^*(D(W^u), F))^*$

Now introduce a Witten–type deformation [38] of our present, non–self adjoint, Laplacian . That is, introduce for each real number t the operators on $A^*(M, F)$

$$\begin{aligned}&\tilde{\Delta}_{T,f,F} = d_{T,f,F}\ \delta_{T,f,F} + \delta_{T,f,F}\ d_{T,f,F} = (\tilde{D}_{T,f,F})^2 \\ &with \\ &\tilde{D}_{T,f,F} = d_{T,f,F} + \delta_{T,f,F} \\ &d_{T,f,F} = e^{-Tf} d_F e^{+Tf}, and,\ \delta_{f,t,F} = e^{Tf} \delta_F^{\#} e^{-Tf}\end{aligned}$$

and define the deformed version of $d_F + \delta^\#_F$.

$$D_{T,f,F} := e^{Tf}\ \tilde{D}_{T,f,F}\ e^{-Tf} = d_F + e^{2Tf}\ \delta^\#_F\ e^{-2Tf}$$

$\tilde{\Delta}_{T,f,F}$ is called a Witten–type Laplacian.

Note that $D_{T,f,F}$ commutes with d_F. Let $\delta^\#_{T,f,F} = e^{2Tf}\delta^\#_F e^{-2Tf}$, so the deformed Laplacian $\Delta^\flat_{T,f,F} := (D_{T,f,F})^2 = d_F\ \delta^\#_{T,f,F} + d_F\ \delta^\#_{T,f,F}$ commutes with d_F. It is differs by a first order operator from $\Delta^\flat_F$, so has all the standard properties.

For $K > 0$ not the real part of any eigenvalue of the elliptic operator $\Delta^\flat_{T,f,F}$, let $V^*(T,f,F,K,g)$ denote the span of the space of generalized eigenforms with real parts less than K, $\Pi^*(T,f,F,K,g)$ the spectral projection on that finite subspace and

$$Ray - Singer - Term(T,f,F,K,g)$$

the Ray and Singer contribution from the eigenvalues with real value greater than K defined as before via zeta function regularization.

Since this deformation merely changes $\hat{\star} d_F \hat{\star}^{-1}$ into $e^{2Tf}\ \hat{\star} d_F \hat{\star}^{-1}\ e^{2Tf}$, the now standard argument shows that the derivative $d/dT\ log[Ray - Singer - Term(T,f,F,K,g)]^2$ is precisely

$$-\Sigma^n_{q=0}\ (-1)^q\ Tr(\Pi^q(T,f,F,K,g)\ \cdot f)$$

The absence of anomaly comes from the fact that M is odd dimensional.

On the other hand, the operator $e^{-2Tf}\hat{\star}$ intertwines the deformed Laplacians for the values $T, -T$:

$$[e^{-2Tf}\ \hat{\star}]\ \Delta^\flat_{T,f,F} = \Delta^\flat_{-T,f,F}[e^{-2Tf}\ \hat{\star}]$$

So the deformed Laplacians $\Delta^\flat_{T,f,F}$ and $\Delta^\flat_{-T,f,F}$ have the same eigenvalues and the induced mapping

$$[e^{-2Tf}\ \hat{\star}] : V^*(T,f,F,K,g) \cong V^{n-*}(-T,f,F,K,g)$$

identical with the above construction for $T = 0$. These subspaces are cochain complexes under d_F which commutes with $\Delta^\flat_{T,f,F}$ and $\Delta^\flat_{-T,f,F}$.

Consequently, one may again form the associated element of $(detH^*(W_u,F))\otimes (det\ H^{n-*}(D(W_u),F))^* \cong \det\ H^*(M,F\oplus F^*)$ as before.

$$\begin{aligned}
&[e^{-2Tf}\ \hat{\star}]^{-1} \in (det\ V^*(T,f,F,K,g)) \otimes (detV^{n-*}(-T,f,F,K,g))^* \\
&\overset{\tau\otimes\tau}{\cong} (det\ H^*(V^*(T,f,F,K,g),d_F) \otimes (detH^{n-*}(V^*(-T,f,F,K,g),d_F))^* \\
&\overset{\int\otimes\int}{\cong}\ (det\ H^*(M,F)) \otimes (det\ H^{n-*}(M,F))^* \cong det\ H^*(M,F\oplus F^*)
\end{aligned}$$

Since e^{-2Tf} merely deforms the star operator, again the now standard methods show that the variation of $(-1)^{S(C)}\ (\int\otimes\int)(\tau\otimes\tau)([e^{-2Tf}\ \hat{\star}]^{-1})$ with T is precisely $+e^{2\Sigma^n_{q=0}\ (-1)^q\ Tr(\Pi^q(T,f,F,K,g)\ \cdot f)}$. This together with the above proves the lemma.

Lemma 9.3 (Invariance under Witten deformation:) *The element* $[Ray - Singer_Term(T, f, F, K, g)]^2 \ (\int \otimes \int)(\tau \otimes \tau)([e^{-2Tf} \ \hat{\star}]^{-1})$ *regarded as an element of det* $H^*(M, F \oplus F^*)$ *is independent of* T*. That is, its equal to* $\tau_{analytic}(W, F, g)$ *for all* T*.*

An alternative formulation of this invariance is that the element

$$\begin{aligned}&[Ray - Singer_Term(T, f, F, K, g)]^2 \quad (\int \otimes \int)(\tau \otimes \tau)([e^{-2Tf} \ \hat{\star}]^{-1})\\ &\in (detH^*(W_u, F)) \otimes (detH^*(D(W_u), F)^*\end{aligned}$$

is independent of T. Here the integrals are carried out over the cells of W_u and the cells of W_s respectively.

To analyze the low eigenvalues of the deformed Laplacian $\Delta^\flat_{T,f,F}$ proceed following Bismut and Zhang and Su and Zhang [8], [35].

For each critical point, say $p \in B$, of index n_p,chose local coordinate $y^1, \cdots, y^n$ so that in this neighborhood U_p

$$f = f(p) - [(1/2)\Sigma_{i=1}^{n_p}(y^i)^2] + (1/2)[\Sigma_{j=n_p+1}^{n} \ (y^j)^2]$$

compatible with the orientation of M. Let W_p^u denote the unstable descending cell from p, and W_p^s the stable ascending cell with their inherited orientations from this choice of coordinates. Take a Hermitain metric on F which is flat on U_p and use the flat sections to identify $F|U_p \cong U_p \times F_p$.

Set $|Y|^2 = \Sigma_{i=1}^{n} \ (y^i)^2$. Let $\gamma(t)$ be non–negative, smooth compactly supported function identically 1 nearby 0 for which $\gamma(|Y|)$ has compact support in the coordinate chart U_p above.

For an element $e \in F_p$ introduce the following Gaussian n_p–form:

$$\begin{aligned}&\rho_{p,T}(e) = \tfrac{\gamma(|Y|)^2}{\sqrt{\alpha_{p,t}}} \ \times exp(-\tfrac{T|Y|^2}{2}) \otimes edy^1 \wedge \cdots \wedge dy^{n_p}\\ &where\\ &\alpha_{p,t} = \int_{U_p} \ \gamma(|Y|)^2 exp(-T|Y|^2) dy^1 dy^2 \cdots dy^n\end{aligned}$$

By sending $[W_p^u] \otimes e$ to $\rho_{p,T}(e) \in A^{n_p}(M, F)$, define a mapping of the descending cochain complex to forms:

$$J_{f,T,u} : C^*(W^u, F) \to A^*(M, F)$$

Similarly, replacing f by $-f$ defines the mapping for the ascending cochain complex:

$$J_{f,T,s} : C^*(W^s, F) \to A^*(M, F)$$

It is a crucial observation that the standard duality isomorphism

$$\Theta : C^a(W^u, F) \cong C^{n-a}(W^s, F)$$

sending $[W_i^u] \otimes e$ to $[W_i^s] \otimes e$ and the star operator

$$\hat{\star} : A^a(M, f) \cong A^{n-a}(M, F)$$

form the commutative diagram:

Lemma 9.4 (Θ and $\star$ compatibiltity)

$$\begin{array}{llcl} J_{T,f,u}: & C^*(W^u,F) & \to & A^*(M,F) \\ & \downarrow \Theta & & \downarrow \hat{\star} \\ J_{T,f,s}: & C^*(W^s,F) & \to & A^*(M,F) \end{array}$$

That is, the geometric duality map Θ becomes precisely the star operator under these two mappings. This observation was emphasized by Bismut and Zhang [8]

Recall the from the definitions, the Witten Laplacian is:

$$\tilde{\Delta}_{T,f,F} := e^{-Tf}\ \Delta_{T,f,F}\ e^{+Tf}$$

It has the property that $\hat{\star}\tilde{\Delta}_{f,T,F} = \hat{\star}\ \tilde{\Delta}_{f,-T,F}$.

Let $A^*_{[0,1],T,f}(M,F)$ be the span of the generalized eigenvectors of $\tilde{\Delta}_{f,T,g}$ with eigenvalues λ with $|\lambda| < 1$. Let $\Pi_{f,T}$ denote the spectral projection of $A^*(M,F)$ onto $A^*_{[0,1],T,f}(M,F)$. Similarly for $f,-T$.

In particular , $a \to e^{tf}a$ carries $A^*_{[0,1],T,f}(M,F)$ isomorphically to $V^*(T,f,F,K=1,g)$.

The basic result is the following theorem, see Bismut and Zhang [8]

Theorem 9.5 *For T sufficiently large there is a $c > 0$ with $||J_{f,T} - \Pi_{T,f}J_{f,T}|| = 0(e^{-cT})$ and $||J_{f,-T} - \Pi_{f,T}J_{f,-T}|| = 0(e^{-cT})$.*

Moreover, $\Pi_{f,T}J_{T,f}, \Pi_{f,-T}J_{f,T}$ are isomorphisms.

Moreover, $\hat{\star}\tilde{\Delta}_{f,T} = \tilde{\Delta}_{t,T}\hat{\star}$ so $\hat{\star}$ induces an isomorphism $A^*_{[0,1],f,T}(M,F) \to A_{[0,1],f,-T}(M,F)$. Also the following commutes

$$\begin{array}{llcl} J_{f,T,u}: & C^*(W^u,F) & \to & A^*_{[0,1],T,f}(M,F) \\ & \downarrow \Theta & & \downarrow \hat{\star} \\ J_{f,T,s}: & C^*(W^s,F) & \to & A^*_{[0,1],T,-f}(M,F) \end{array}$$

These relations can be explored using the commutative diagram with all horizontal arrows isomorphisms for T sufficiently large,

$$\begin{array}{ccccccc} C^*(W^u,F) & \to & A^*_{[0,1],f,T}(M,F) & \overset{\times e^{Tf}}{\to} & V^*(f,T,F,K,g) & \overset{\int}{\to} & C^*(W_u,F) \\ \downarrow \Theta & & \downarrow \hat{\star} & & \downarrow e^{-2Tf}\hat{\star} & & \downarrow \\ C^*(W^s,F) & \to & A^*_{[0,1],f,-T}(M,F) & \overset{\times e^{-Tf}}{\to} & V^*(f,-T,F,K,g) & \overset{\int}{\to} & C^*(W_s,F) \end{array}$$

where the first maps are the isomorphisms: $\Pi_{f,T}\ J_{f,T,u} : C^*(W^u,F) \to A^*_{[0,1],f,T}(M,F)$ and $\Pi_{f,-T}J_{f,T,s} : C^*(W^s,F) \to A^*_{[0,1],f,-T}(M,F)$.

By the above, the algebraic torsion contribution to $\tau_{analytic}(M,f,T,g)$ of the eigenspace $V^*(T,f,F,K,g)$ and its two differentials is the same as taking the

isomorphism $[e^{-2Tf}\hat{\star}]^{-1}$, regarding it as an element of $(det\ V^*(f,T,F,K,g)) \otimes (det\ V^{n-*}(f,-T,F,K,g))^*$ and applying the integration mapping to send it to $(det\ C^*(W^u,F)) \otimes (det\ C^{n-*}(W^s,F))^*$. By this commutative diagram this may be done by starting with the element $[\Theta]^{-1}$ as an element of $(det\ C^*(W^s,F)) \otimes (det\ C^{n-*}(W^s,F))^*$ and passing it along from the left to the right.

Now by theorem 9.5 this composite is approximated with error (e^{-ct}) by the composite coming from modified diagram:

$$\begin{array}{ccccccc} C^*(W^u,F) & \stackrel{J_{f,T,u}}{\rightarrow} & A^*(M,F) & \stackrel{\times e^{Tf}}{\rightarrow} & A^*(M,F) & \stackrel{\int}{\rightarrow} & C^*(W_u,F) \\ \downarrow \Theta & & \downarrow \hat{\star} & & \downarrow e^{-2Tf}\hat{\star} & & \downarrow \\ C^*(W^s,F) & \stackrel{J_{f,T,u}}{\rightarrow} & A^*(M,F) & \stackrel{\times e^{Tf}}{\rightarrow} & A^*(M,F) & \stackrel{\int}{\rightarrow} & C^*(W_s,F) \end{array}$$

This computing these integrals yields the following lemma. It is the analogue of Theorem 3.3 of the paper of Su and Zhang.

Lemma 9.6

$$\lim_{T\to+\infty} \frac{\tau(e^{Tf}\ V(K=1,F,g), d_F, \hat{\star}\ d_F\ \hat{\star})}{\tau(C^*(W^u,F), d_F, \Theta^{-1}d_F\Theta)} \cdot (\tfrac{T}{\pi})^{((n/2)r-s)} exp(2k\Sigma_p\ (-1)^{n_p}\ f(p))$$

$$= 1$$

with $r = k(\chi(M)) = k(\Sigma_p\ (-1)^{n_p}), s = k(\Sigma_p\ (-1)^{n_p} f(p))$.

Combining the Witten invariance lemma 9.3 with this last lemma, it follows that theorem 9.2 and so theorem 9.1 will be established once it is proved that

$$lim_{T\to\ +\infty}\ [(\tfrac{T}{\pi})^{((n/2)r-s)} exp(2k\Sigma_p\ (-1)^{n_p}\ f(p))]^{-1}$$
$$(Ray-Singer-Term(T,f,F,K=1,g))^2 = 1$$

Here the eigenvalues can be taken as those of the Witten Laplacian $\tilde{\Delta}_{f,T,g}$ which has the same eigenvalues as $\Delta_{f<T,g}$

In the paper of Su and Zhang [35] the analogous result is proved for the Witten version of the operator of Haller and Burghelea. Fortunately their methods are very robust and apply without essential change to the Witten–type Laplacian above. The only major change is the above control on the small eigenvalues. This concludes the proof of the generalized Cheeger Múller result 9.1.

10 Zeta functions for the d-bar setting:

The analytical details are given in the context of the $\bar{\partial}$ Laplacian coupled to a holomorphic bundle E with compatible type $(1,1)$ connection. The analogous results for the Riemannian Laplacian coupled to a flat complex bundle follow the same outline, so their proofs are omitted. All this material is relatively standard

and is often used without comment in the literature. The basic reference is Ray and Singer [30, 31].

Let E be a holomorphic bundle with compatible connection D of type $(1,1)$ over a complex Hermitian manifold W. Let a Hermitian metric $< .,. >_E$ be chosen for E and the adjoint of the d-bar operator under the induced inner product on $A^{p,q}(W,E)$ be denote by $\delta_E = adjoint(\bar{\partial}_E)$. Let the associated self–adjoint d-bar Laplacian be denoted by Δ_E.

$$\Delta_E = (\bar{\partial}_E + adjoint(\bar{\partial}_E))^2 = (\bar{\partial}_E + \delta_E)^2 = \bar{\partial}_E\ \delta_E + \delta_E\ \bar{\partial}_E$$

In this section fix $p, 1 \leq p \leq N$ and throughout consider all operators acting on $A^{p,q}(W,E)$ for p fixed with possibly varying q.

Recall from equation 4.15 , $\bar{\partial}^*_{E,D''} = \delta_E + \alpha$ for a smooth bundle mapping $\alpha : \Lambda^q(T^*M) \to \Lambda^{q-1}(T^*M)$. Moreover, $\Delta_{E,\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*_{E,D''})^2 = (\ \bar{\partial}_E + adjoint(\bar{\partial}_E))^2 + \alpha\bar{\partial}_E + \bar{\partial}_E\alpha = \Delta_E + \alpha\bar{\partial}_E + \bar{\partial}_E\alpha$ hence is a elliptic second order partial differential equation with scalar symbol.

Proof of lemma 4.1:

Use the reference choice of Hermitian inner product $< .,. >_E$ on E to define all norms, Solobev spaces, etc. This is used to construct the adjoint δ_E and the self-adjoint d-bar Laplacian Δ_E as above.

Since $G_t = (\lambda I - (\bar{\partial}_E + \delta_E + t\alpha)), 0 \leq t \leq 1,$ is a first–order elliptic operator, it is Fredholm regarded as a map of Sobolev spaces $A^{p,*}(W,E)_{(1)} \to A^{p,*}(W,E)_{(0)}$, losing one derivative.

As G_0 is self-adjoint, it has index zero. Hence by homotopy invariance of the index, $G_1 = (\lambda I - (\bar{\partial} + \bar{\partial}^*_{E,D''}))$ is Fredholm of index zero. In particular, if one shows that the kernel, $ker\ G_1$, is zero, then G_1 is also onto and so an isomorphism. That is, λ will be not be in the spectrum of $(\bar{\partial}_E + \delta_E + \alpha) = \bar{\partial} + \bar{\partial}^*_{E,D''}$ unless there is an eigenvector with eigenvalue λ.

Since $\bar{\partial}_E + \delta_E$ is self-adjoint, we may chose an orthonormal basis, say $\{\phi_j,\ j = 1, \cdots\}$, of eigenvectors for the L^2 completion of $A^*(M,E)$. Let $(\bar{\partial}_E + \delta_E)\phi_j = \mu_j\ \phi_j$ for real non-negative numbers μ_j,

Given $a = \Sigma_j a_j \phi_j$ with $||a||^2 = \Sigma_j\ ||a_j||^2 = 1$, one has the simple estimates $||(\lambda I - (\bar{\partial}_E + \bar{\partial}^*_{E,D''}))a|| = ||(\lambda I - (\bar{\partial}_E + \delta_E + \alpha)a|| \geq ||(\lambda I - (\bar{\partial}_E + \delta_E))a|| - ||\alpha a|| = ||\Sigma_j\ (\lambda - \mu_j)a_j\phi_j|| - ||\alpha a|| \geq \sqrt{\Sigma_j[(Im(\lambda))^2 + (Re(\lambda - \mu_j))^2]||a_j||^2} - ||\alpha||\ ||a|| \geq |Im(\lambda)| - ||\alpha||$ because the eigenvalues μ_j are real. Hence, if $|Im(\lambda)| > ||\alpha||$, then necessarily $(\lambda I - (\bar{\partial}_E + \bar{\partial}^*_{E,D''}))a \neq 0$. This also implies the claim since $\Delta_{E,\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*_{E,D''})^2$. The parabola described in lemma 4.1 is the image under $z \mapsto z^2$ of the lines $Im\ z = \pm||\alpha||$.

The analytic continuation of the zeta function is addressed as follows. Let $\Pi_{E,K,g}$ denote the projection onto the span of the generalized eigenspaces with eigenvalues with real parts less than $K > 0$. Set $Q(E,K,g) = (1 - \Pi(E,K,g))$ and $\Delta_{E,K,\bar{\partial},q} = Q(E,K,g)\Delta_{E,\bar{\partial}}$ acting on (p,q) forms, intuitively this is the heat kernel projected onto the subspace of generalized eigenspaces with real parts at least K. By the estimates of Seeley [33], the associated heat kernel is

most concretely and explicitly given by the contour integral:

$$e^{-\Delta_{E,K,\partial,q}\,t} \overset{definition}{:=} \frac{1}{2\pi i}\int_{\gamma_K} e^{-\lambda t}\ (\lambda - \Delta_{E,K,\partial,q})^{-1}\ d\lambda$$

where γ_K is is the parabola containing all the spectrum given by $X = A\,Y^2 - B$ with $B > ||\alpha||^2$ and $0 < A < (1/(4||\alpha||^2)$ traced counterclockwise.

One assumes that $K > 0$ is not the real part of the any generalized eigenvalue of $\Delta_{E,\partial,q}$, for any q, then $e^{-\Delta_{E,K,\partial,q}\,t}$ is well defined, trace class for $t > 0$, and moreover,

$$Tr(e^{-\Delta_{E,K,\partial,q},t}) = \Sigma_j\ e^{-\ \lambda_{q,K,j}\ t} \quad \text{for } t > 0,$$

where the sum is over the generalized eigenvalues of $\Delta_{E,\partial}$ acting on (p,q) forms counted with multiplicities with real part greater than K.

Let $\{\nu(q,K,1),\cdots,\nu(q,K,N(q,K))\}$ be a complete enumeration counted with multiplicities of the finitely many generalized eigenvalues of $\Delta_{E,\partial}$ having real parts less than K. Tautologically one has:

$$\Sigma_{j=1}^{N(q,K)}\ e^{-\nu(q,K,j)\ t} + Tr(e^{-\Delta_{E,K,\partial,q}\,t}) = Tr(e^{-\Delta_{E,\partial,q}\,t})$$

By Seeley's results the short time asymptotics of the trace of the complete heat kernel is of the form

$$Tr(e^{-\Delta_{E,\partial,q}\,t}) = \Sigma_{j=0}^{M}\ t^{-n/2+j} a_{q,j} + 0(t^{-n/2+M+1})$$

where the $a_{q,j} = \int_W\ b_{q,,j,u}$ are constructed out of integrals of locally constructed forms $b_{q,j,u}$ depending only on finitely many covariant derivatives of the Hermitian metric g on TW and the connection form of D. Recall $n = dim\ W = 2N$.

Expressed another way, with

$$Tr(e^{-\Delta_{E,K,\partial,q}\,t}) + (\Sigma_{j=1}^{N(q,K)}\ e^{-\nu(q,K,j)\ t}) - (\Sigma_{j=0}^{N}\ t^{n/2+j} a_{q,j}) = \mu_M(t)$$

where $\mu_M(t) = 0(t^{-n/2+M+1})$.

Another implication of Seeley's analysis is control on the rate of growth of the real parts of the eigenvalues, which give uniform estimates

$$Tr(e^{-\Delta^{E,\bar\partial,K,q}\,t}) \leq Ce^{-Dt}\ for\ t > 1\ ,$$

for some constants $C > 0, D > 0$. Also one gets convergence of the zeta function:

$$\zeta_{q,K}(s) := \Sigma_j \frac{1}{(\lambda_{q,K,j})^{-s}},\ for\ Re(s) > n/2\ .$$

These facts may be used to give an explicit formula for the analytic continuation of this zeta function to $s = 0$ and its derivative at zero, as in Cheeger's paper [19].

First one has for $Re(s) > n/2$ the convergent representation:

$$\zeta_{q,K}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \; t^{s-1} Tr(e^{-\Delta_{E,\bar{\partial},K,q}\, t}) \; dt$$

Taking $M > n/2+1$, choosing $\epsilon > 0$, replacing $\frac{1}{\Gamma(s)}$ by $\frac{s}{\Gamma(s+1)}$ and substituting, one has equivalently:

$$\begin{aligned}
&\zeta_{q,K}(s) = \tfrac{s}{\Gamma(s+1)} \int_0^\epsilon \; t^{s-1} \; \mu_M(t) \; dt \\
&-\tfrac{s}{\Gamma(s+1)} \int_0^\epsilon \; t^{s-1} \; (\Sigma_{j=1}^{N(q,K)} \; e^{-\nu(q,K,j) \; t}) \; dt \\
&\quad +\tfrac{s}{\Gamma(s+1)} \int_0^\epsilon \; t^{s-1} \; (\Sigma_{j=0}^{M} \; t^{-n/2+j} a_{q,j}) \; dt \\
&\qquad +\tfrac{s}{\Gamma(s+1)} \int_\epsilon^\infty \; t^{s-1} Tr(e^{-\Delta_{E,\bar{\partial},K,q}\, t}) \; dt
\end{aligned}$$

By $\Gamma(1) = 1, \Gamma'(1) = \gamma$, and $M > n/2+1$, the first term has analytic continuation to $Re(s) > -1$ by this same convergent expression. Its derivative by s is, $(\frac{1}{\Gamma(s+1)} - \frac{s}{\Gamma(s+1)^2}(\Gamma'(s+1))) \int_0^\epsilon \; t^{s-1} \; \mu_N(t) \; dt + \frac{s}{\Gamma(s+1)} \int_0^\epsilon \; (log(t) t^{s-1}) \; \mu_K(t) \; dt$. Evaluated at $s=0$ this yields $\int_0^\epsilon \; t^{-1} \; \mu_K(t) \; dt$ which converges.

The second term once $N(q,K)$ is added is similar, $-\frac{s}{\Gamma(s+1)} \int_0^\epsilon \; t^{s-1}$ $(\Sigma_{j=1}^{N(q,K)} \; (e^{-\nu(q,K,j) \; t} - 1)) \; dt$ since $e^{-\nu(q,K,j) \; t} - 1 = 0(t)$, converges for $Re(s) > -1$. The derivative evaluated at $s=0$ yields $-\int_0^\epsilon \; t^{-1} \; (\Sigma_{j=1}^{N(q,K)} \; (e^{-\nu(q,K,j) \; t} - 1)) \; dt$ which converges.

The canceling term is $-N(q,K) \; \frac{s}{\Gamma(s+1)} \int_0^\epsilon \; t^{s-1} \cdot 1 \; dt = -N(q,K) \; \frac{s}{\Gamma(s+1)} \frac{\epsilon^s}{s} = -N(q,K) \; \frac{\epsilon^s}{\Gamma(s+1)}$. This expression is analytic for $Re(s) > -1$ and has derivative at $s=0$ equal to $-N(q,K)(log(\epsilon) - \gamma)$.

In the third term, omitting the constant term t^0, i.e. omitting $j = n/2$, can be integrated, $\frac{s}{\Gamma(s+1)} \int_0^\epsilon \; t^{s-1} \; (\Sigma_{j=0, j\neq n/2}^{M} \; t^{-n/2+j} a_{q,j}) \; dt = \frac{s}{\Gamma(s+1)} \times$ $(\Sigma_{j=0, j\neq n/2}^{M} \; \frac{\epsilon^{-n/2+j+s}}{-n/2+j+s} a_{q,j})$. It analytically extends to $s=0$ and has derivative $(\frac{1}{\Gamma(s+1)} - \frac{s}{\Gamma(s+1)^2}(\Gamma'(s+1))) \times (\Sigma_{j=0, j\neq n/2}^{M} \; \frac{\epsilon^{-n/2+j+s}}{-n/2+j+s} a_{q,j}) + \frac{s}{\Gamma(s+1)} (\Sigma_{j=1}^{N} \; \frac{\epsilon^{-n/2+j+s}}{-n/2+j+s} a_j) \times$

$log(-n/2+j+s)$. Evaluated at $s=0$ this is $(\Sigma_{j=0, j\neq n/2}^{M} \; \frac{\epsilon^{-n/2+j}}{-n/2+j} a_{q,j})$.

The term $a_{n/2,q} \; t^0$ contributes to the zeta function at zero the term $a_{n/2,q} \; (log(\epsilon) - \gamma)$.

The last term is uniformly convergent, extends by this formula to $s=0$ and has at that point the derivative $\int_\epsilon^\infty \; t^{-1} \; Tr(e^{-\Delta_{q,K}\, t}) \; dt$, similar to the first term.

In toto, the analytic continuation of $\zeta_{q,K}(s)$ to $s=0$ has the derivative at $s=0$ computed as above. That is,

Theorem 10.1 *For the notation above,*

$$\begin{aligned}
&\zeta'_{q,K}(0) = \int_0^\epsilon \; t^{-1} \; \mu_M(t) \; dt \\
&-\int_0^\epsilon \; t^{-1} \; (\Sigma_{j=1}^{N(q,K)} \; (e^{-\nu(q,K,j) \; t} - 1)) \; dt \\
&+(a_{n/2,q} - N(q,K))(log(\epsilon) - \gamma) + (\Sigma_{j=0, j\neq n/2}^{M} \; \tfrac{\epsilon^{-n/2+j}}{-n/2+j} a_{q,j}) \\
&+\int_\epsilon^\infty \; t^{-1} \; Tr(e^{-\Delta_{E,\bar{\partial},K,q}}) \; dt.
\end{aligned}$$

Let $0 < K < L$ be chosen so that $\Delta_{E,\bar{\partial},q}$ acting on $A^{p,q}(W,E)$ for any q has no generalized eigenvalue with real part equal to K or L. Let $\lambda_k, k = 1, \cdots, T$ be a complete enumeration, counting with multiplicities, of the generalized eigenvalues of $\Delta_q^\flat$ with real parts between K and L. Denote by $log(\lambda_k)$ the principle value of the eigenvalue λ_k.

Then

$$\zeta'_{q,L}(0) - \zeta'_{q,K}(0) = -\Sigma_{k=1}^T \; log(\lambda_k) \tag{10.67 }$$

Note that this theorem implies lemma 4.3.

Proof of the equation 10.67 : The formula for $\zeta'_{q,L}(0)$ and $\zeta'_{q,K}(0)$ just derived may be subtracted. The terms involving $\mu_M(t)$ and $a_{q,j}$ cancel, yielding

$$\begin{aligned}
&\zeta'_{q,L}(0) - \zeta'_{q,K}(0) = -\int_0^\epsilon \; t^{-1} \; (\Sigma_{k=1}^T \; (e^{-\lambda_k \; t} - 1)) \; dt \\
&-T(log(\epsilon) - \gamma) + \int_\epsilon^\infty \; t^{-1} \; (\Sigma_{k=1}^T \; e^{-\lambda_k t}) \; dt \\
&= \Sigma_{k=1}^T \; [-\int_0^\epsilon \; t^{-1} \; (e^{-\lambda_k \; t} - 1)) \; dt - (log(\epsilon) - \gamma) \\
&+ \int_\epsilon^\infty \; t^{-1} \; e^{-\lambda_k t} \; dt] = -\Sigma_k \; \log(\lambda_k)
\end{aligned}$$

To see the last equality, recall that for $Re(\lambda) > 0$, one has taking principle values $\frac{1}{\lambda^s}$ analytic with derivative $\log(\lambda)(\frac{1}{\lambda^s})$ which evaluates at $s = 0$ to $-log(\lambda)$.

On the other hand there is also the convergent integral expression $\frac{1}{\lambda^s} = \frac{s}{\Gamma(s+1)} \int_0^\infty \; t^{s-1} Tr(e^{-\lambda t}) \; dt$ which by the above analysis has analytic continuation to $s = 0$ with derivative given by $[-\int_0^\epsilon \; t^{-1} \; (e^{-\lambda_k \; t} - 1)) \; dt - (log(\epsilon) - \gamma) + \int_\epsilon^\infty \; t^{-1} \; e^{-\lambda_k t} \; dt]$. Hence, these are equal.

Let $g(u), a \leq u \leq b$ be varying Hermitian metrics on the complex manifold W. Suppose that $K > 0$ is not an generalized eigenvalue for any of the $\Delta_{E,\bar{\partial},q,g(u)}$ acting on $A^{p,q}(W,E)$ for any q and any u. By the formulas for the spectral projections $\Pi_{E,K,q,g(u)}$, the modified heat kernels, $e^{-\Delta_{E,\bar{\partial},K,q,g(u)} \; t}$, vary smoothly with u and one may address the question of computing the variation of the analytic continuation $\zeta'_{q,K,g(u)}(0)$ with the metric. Here one explicitly records the metric dependence.

The foundational initial formula, as in Ray and Singer [30], is:

Lemma 10.2

$$\begin{aligned}
&d/du \; \Sigma_{q=0}^N \; (-1)^q q \; Tr[e^{-\Delta_{E,\bar{\partial},K,q,g(u)} t}] \\
&= \; t \; d/dt[\Sigma_{q=0}^N \; (-1)^q \; Tr(e^{-\Delta_{E,\bar{\partial},K,q,g(u)} \; t} \quad \alpha_{u,p,q})]
\end{aligned}$$

where $\alpha_{u,p,q} = (\star_{u,p,q})^{-1} \; d/du \; (\star_{u,p,q}) = (\hat{\star}_{t,p,q})^{-1} \; d/du \; (\hat{\star}_{t,p,q})$ acting on $A^{p,q}(W,E)$.

Proof of lemma 10.2: Note that spectral projection $\Pi_{E,K,q,g(u)} = (\Pi_{E,K,q,g(u)})^2$ has finite rank, and $Q_{E,K,q,g(u)} = 1 - \Pi_{E,K,q,g(u)}$ is also a projection so $(Q_{E,K,q,g(u)})^2 = Q_{E,K,q,g(u)}$. In particular, $d/du \, Q_{E,K,q,g(u)} = -d/du \, \Pi_{E,K,q,g(u)} =$

$-d/du\,(\Pi_{E,K,q,g(u)})^2 = -d/du\,\Pi_{E,K,q,g(u)}\,\Pi_{E,K,q,g(u)} - \Pi_{E,K,q,g(u)} d/du\,\Pi_{E,K,q,g(u)}$ is of finite rank and consequently is of trace class.

Differentiating $(Q_{E,K,q,g(u)})^2 = Q_{E,K,q,g(u)}$ yields $Q_{E,K,q,g(u)}\ d/du\ (Q_{E,K,q,g(u)}) + d/du\ (Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)} = d/du\ (Q_{E,K,q,g(u)})$. Applying $Q_{E,K,q,g(u)}$ to each side gives $2\ Q_{E,K,q,g(u)}\ d/du\ (Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)}$ $=\ Q_{E,K,q,g(u)}\ d/du\ (Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)}$. Hence,

$$Q_{E,K,q,g(u)}\ d/du\ (Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)} = 0$$

In particular, $d/du\,(Tr(e^{-\Delta_{E,\bar\partial,K,q,g(u),t}})) = d/du(Tr(Q_{E,K,q,g(u)}\ e^{-\Delta_{E,\bar\partial}\ t})) = Tr([d/du\ Q_{E,K,q,g(u)}]\ \ e^{-\Delta_{E,\bar\partial}\ t}) + Tr(Q_{E,K,q,g(u)}\ d/du(e^{-\Delta_{E,\bar\partial}\ t}))$ $= Tr([Q_{E,K,q,g(u)}\ d/du\,(Q_{E,K,q,g(u)}) + d/du\,(Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)}]\ \ e^{-\Delta_{E,\bar\partial}\ t}) + Tr(Q_{E,K,q,g(u)}\ d/du(e^{-\Delta_{E,\bar\partial}\ t})$. On the other hand, by the above since $Q_{E,K,q,g(u)}$ commutes with $e^{-\Delta_{\bar\partial,q,g(u)}\ t}$, $Tr([Q_{E,K,q,g(u)}\ d/du\ (Q_{E,K,q,g(u)})\ e^{-\Delta_{E,\bar\partial,g(u)}\ t}) = Tr([(Q_{E,K,q,g(u)})^2\ d/du\ (Q_{E,K,q,g(u)})\ e^{-\Delta_{E,\bar\partial,g(u)}\ t}) = Tr([Q_{E,K,q,g(u)}$ $d/du\ (Q_{E,K,q,g(u)})\ e^{-\Delta_{E,\bar\partial,g(u)}\ t}\ Q_{E,K,q,g(u)})$ $= Tr([Q_{E,K,q,g(u)}\ d/du\,(Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)}\ e^{-\Delta_{E,\bar\partial,g(u)}\ t}\ ) = 0$ and similarly $Tr(\ d/du\ (Q_{E,K,q,g(u)})\ Q_{E,K,q,g(u)}\ \ \ e^{-\Delta_{E,\bar\partial,g(u)}\ t}) = 0$. These in combination with the above prove the following formula:

$$d/du\ (Tr(e^{-\Delta_{E,\bar\partial,K,q,g(u)} t})) = Tr(Q_{E,K,q,g(u)}\ d/du(e^{-\Delta_{E,\bar\partial,g(u)}\ t}))$$

Now the methods of Ray and Singer in either of [30, 31] apply verbatim to complete the proof of lemma 10.2.

Theorem 10.3 *Under the assumptions above, the analytic continuation of* $\zeta_{q,K}(s)$ *to* $s = 0$ *has the property:*

$$d/du\ [\Sigma_q\ (-1)^q q\ \zeta'_{q,K}(0)] = \Sigma_{q=0}^n\ (-1)^q\ Tr(\Pi_{q,K}\ \ \cdot\alpha) + a_{n/2,p,q}$$

where the prime $'$ *denotes the derivative and* $\alpha = \star^{-1}\ d/du(\star)$.

Following Ray and Singer, a consequence of lemma 10.2 is a formula for the variation of our desired quantity, for $Re(s) > n/2$:

$$\begin{aligned}
&d/du[\Sigma_q\ (-1)^q q\ \zeta_{q,K}(s)] \\
&= d/du\ [\Sigma_q (-1)^q q\ \tfrac{1}{\Gamma(s)} \textstyle\int_0^\infty\ t^{s-1} Tr(e^{-\Delta_{E,\bar\partial,K,q,g(u)}\ t}\ dt] \\
&= \tfrac{1}{\Gamma(s)} \textstyle\int_0^\infty\ t^s\ d/dt[\Sigma_q\ (-1)^q\ Tr(e^{-\Delta_{E,\bar\partial,K,q,g(u)}\ t}\cdot\alpha)]\ dt \\
&= -\tfrac{s^2}{\Gamma(s+1)}\Sigma_q\ (-1)^q\ \textstyle\int_0^\infty\ t^{s-1}\ Tr(e^{-\Delta_{E,\bar\partial,K,q,g(u)}\ t}\cdot\alpha)\ dt \\
&= -\tfrac{s^2}{\Gamma(s+1)}\Sigma_q\ (-1)^q\ \textstyle\int_0^\infty\ t^{s-1}\ Tr(e^{-\Delta_{E,\bar\partial,q,g(u)}\ ,t}\cdot\alpha \\
&-\Pi_{E,K,q}\ e^{-\Delta_{E,\bar\partial,q,g(u)}\ ,t}\cdot\alpha)\ dt \\
&= -\tfrac{s^2}{\Gamma(s+1)}\Sigma_q\ (-1)^q\ \textstyle\int_0^\infty\ t^{s-1}\ Tr(e^{-\Delta_{E,\bar\partial,q,g(u)}\, t}\cdot\alpha)\ dt \\
&+\tfrac{s^2}{\Gamma(s+1)}\Sigma_q\ (-1)^q\ \textstyle\int_0^\infty\ t^{s-1}\ Tr(-\Pi_{E,K,q}\ (e^{-\Delta_{E,\bar\partial,q,g(u)}\, t} - Id)\cdot\alpha)\ dt \\
&-\tfrac{s^2}{\Gamma(s+1)}\Sigma_q\ (-1)^q\ \textstyle\int_0^\infty\ t^{s-1}\ Tr(\Pi_{E,K,q}\ \ \cdot\alpha)\ dt
\end{aligned}$$

Now $n = dim\ W = 2N$ is even, so the asymptotic expansion for $Tr(e^{-\Delta_{E,\bar{\partial},q}}\ \alpha_{E,p,q,u})$ for small time has a term $a_{n/2,p,q,u}\ t^0$ in its expansion about $t = 0$ and the rest positive and finitely many negative powers of t. Then inserting this shows that $\int_\epsilon^\infty\ t^{s-1}\ Tr(e^{-\Delta_{E,\bar{\partial},q}\ t}\ \alpha_{E,p,q,u}))\ dt$ $+ \int_0^\epsilon\ t^{s-1}\ [Tr(e^{-\Delta_{E,\bar{\partial},q}\ t}\ \alpha_{E,p,q,u}) - a_{n/2,p,q,u}\ t^0])\ dt$ extends as a holomorphic function to $s = 0$, so the derivative of the analytic continuation of the first term at $s = 0$ vanishes. Similar remarks hold for the term involving $Tr(-\Pi_{q,K,g(u)}\ (e^{-\Delta_{E,\bar{\partial},q}\,t} - Id) \cdot \alpha_{E,p,q,u})$.

The integral in the last term has a pole of order one with residue $Tr(\Pi_{q,K,g(u)} \cdot \alpha_{E,p,q,u})$, so the analytic continuation of this term to zero exists and has derivative $Tr(\Pi_{q,K,g(u)}\ \cdot \alpha_{E,p,q,u})$. Similarly, the integral for $\int_0^\epsilon\ t^{s-1}\ a_{n/2,p,q}\ t^0\ dt$ has a pole of order 1 with residue $a_{n/2,p,q}$, so the analytic continuation of this term to zero exists and has derivative $a_{n/2,p,q}$. In toto, this proves the theorem 10.3.

This theorem 10.3 is recorded as lemma 7.1.

The completely parallel analysis for the flat Laplacian $\Delta_E^\flat$, yields the corresponding results in this setting. In combination with the analogue of the variation theorem 4.4, which is proved by a completely parallel analysis, the independence of metric result, 7.3, follows.

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